To appear in the Proceedings of the 1999 CMESG Conference

The Decline and Rise of Geometry in 20th Century North America.

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Introduction

While I will begin with my own evidence for the decline of geometry in this century and my own description on how such a decline has proceeded, my basic theme is hopeful. Geometry has not died because it is essential to many other human activities and because it is so deeply embodied in how humans think. With the introduction of computers with rich graphical capacities and the recognition of multiple ways of learning, our current situation offers an unprecedented opportunity for geometers and those who work visually. An independent, but related, description of this past decline and the present possibilities can be found in [55].

I. The Decline of Geometry through the 20th Century

As a graduate student, I worked in an area of mathematics that was officially 'dead': invariant theory or a classical theory of the foundations of analytic geometry [18, 19]. From conversations with mathematicians and from reading the sociology of mathematics, I learned that a field of mathematics 'dies' when it is no longer viewed as an 'important' area of mathematical research. Geometry 'died' in this sense by the mid 20th century in North America. I now see that geometry in the education system then followed in a predictable (though not inevitable) decline. This decline proceeded from the graduate schools into the high school and elementary classrooms over the last decades. Knowing this path may help us plan tactics and strategies for accelerating the rise of geometry. We do not have a half-century to spare for a comparable, gradual 'rise' of geometry!

Let me begin with an 'indicator' that discrete geometry has declined as an 'important area' of mathematical research. At the turn of this century, David Hilbert delivered a famous lecture containing twenty-three problems that might shape mathematics in the 20th century [7]. How many of these were problems in discrete geometry? Three out of twenty three - about 13% of the problems! Hilbert also expressed his sense of geometry in the very readable book [26]. In 1976, a symposium was held on the mathematics arising from these problems [7]. A group of mathematicians gave twenty eight sets of problems - and none of these sets included discrete geometry (or its relatives, such as combinatorics) - although there were sets of problems in more 'current' geometry: algebraic geometry, differential geometry, geometric topology etc. Discrete geometry was no longer an important area. In case you think this was because all the problems were solved, an important part of Hilbert's eighteenth problem: proving that packing spheres in three space like the standard packing of oranges, is the best possible; was solved by Thomas Halles in 1998. See [13] for a recent collection of unsolved problems in discrete geometry.

Philip Davis has chronicled the rise fall of a specific field of discrete geometry – 'triangle geometry' - over the 19th and 20th centuries [14]. Within a richer analysis of the sociology of this decline in the very geometry most often taught in high schools, he quotes E.T. Bell [5 page 323] :

> " The geometers of the 20th century have long since piously removed all these treasures to the museum of geometry where the dust of history quickly dimmed their luster."

To summarize, my preliminary point is that discrete geometry virtually died as an 'important' field of mathematical research through the twenties and thirties and forties, at least in North America and parts of Europe. It survived in pockets (Hungary, Germany, Switzerland, Austria, Russia…) and through a few key people in other places (H.S.M. Coxeter, D. Pedoe, B. Grünbaum). In the Canadian context, this death was confirmed as Professor Coxeter retired at the University of

Toronto several decades ago. The department followed a policy of not hiring in discrete geometry and shifted to the 'hotter' areas such as algebraic geometry. Here is a visual representation for this decline of discrete geometry as a field of research (Figure 1A).

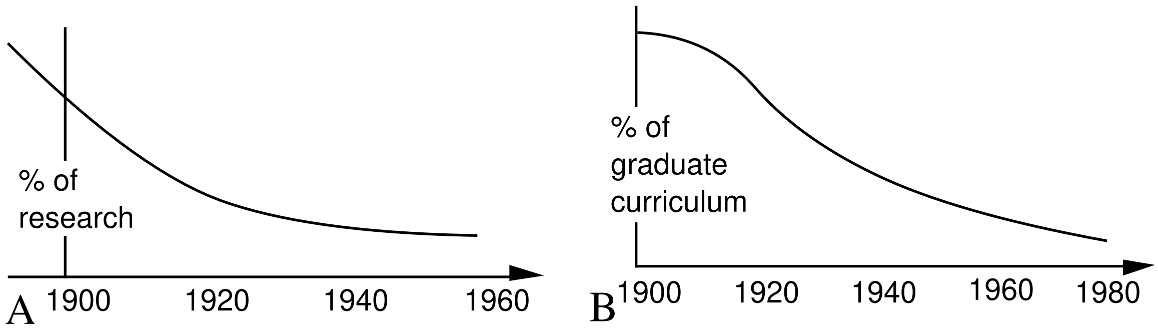

Figure 1: Geometry in decline: research (A) and graduate programs (B).

Here is my model of how this decline was transmitted down from 'research activities' to various levels of mathematics education. As research in geometry declined, the importance of teaching geometry in graduate programs also declined, as did the number of faculty proposing courses in geometry. More and more graduate programs contained no researchers in geometry. No graduate courses in geometry were taught - or if taught the topics were not on the core syllabus or comprehensive exams. Of course, there is lag in this and the previous curve of decline shifts over several decades (Figure 1B)

After a few decades more, we have a generation of people moving out to teach undergraduate mathematics who have not experienced discrete geometry as an important, lively field of current mathematics, and who may not have studied any geometry during their graduate studies. If geometry is then taught to undergraduates, it is taught by someone who is not a geometer and who does not work with visual forms - often by a logician or an historian of mathematics. As a whole, both of these groups would teach geometry as an important past accomplishment (often as an axiomatic study and an exercise in logical proofs) but not as a continuing source of new mathematics. Many undergraduate geometry courses wear a veneer of geometric language without any playful geometric and visual spirit in the problems, the solutions, or the presentation. Over time, this decline reached the point where algebra and analysis became the core areas in the undergraduate curriculum. Geometry was relegated to a service course for future high school teachers by the sixties and perhaps not even that by the nineties. So the curve of decline has shifted over again (Figure 2A).

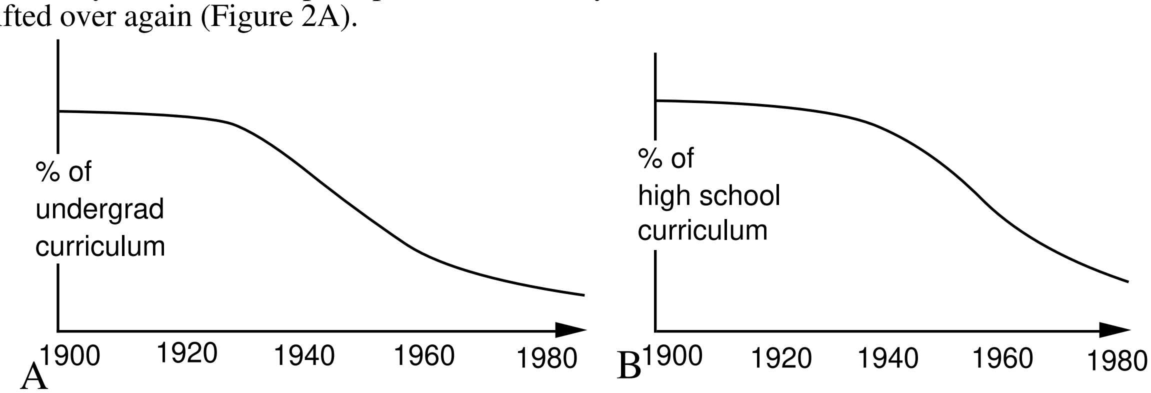

Figure 2. Geometry in decline: undergraduate (A) and high school curriculum (B).

After a few more decades, we have a generation of high school teachers who either had no geometry among their undergraduate courses or had a 'course for teachers'. This implicitly communicates that geometry is not a central part of modern mathematics. Consider the two questions I asked during my talk:

(i) How many of your had an undergraduate geometry course?
(ii) How many Faculties of Education require a course in geometry vs. requiring a course in calculus, linear algebra or statistics?

A few years ago, a group of graduating pre-service students in a geometry course asked me: "Why do we teach geometry in high school?" After some reflection, I realized what their question was about. They took geometry in high school but did not see any material that connected with it during their previous undergraduate program! They were asking: Why teach something in high school that is a 'dead end' for learning more mathematics?

The final stage of this decline is a group of teachers who may be uncomfortable with open-ended problems in geometry and who will leave geometry 'to the end' as something that is much less important to their students than core areas like functions, algebra, calculus.... The sense that geometry is an 'optional' topic continues to grow among the curriculum writers, the textbook writers, the tutors and the parents. The shift to the 'new math', with its emphasis on set theory and algebra, encoded this decline in geometry. The message that geometry is not important is embedded in the dominant culture in undergraduate mathematics departments and in high school mathematics curricula in North America today [46 p.184]. This fits a final shift in the curve of decline (Figure 2B).

This identification of mathematics with language and formulae is also characteristic of people working in the foundations of mathematics [8] and, to a significant extent, of people in algebra and analysis. For example, this is explicit in the Bourbaki tradition. For example, Dieudonné urges a "strict adherence to the axiomatic methods, with no appeal to the "geometric intuition", at least in formal proofs: a necessity which we have emphasized by deliberately abstaining from introducing any diagram in the book" [8 pp. 173-174]. I recall that my linear algebra text had one almost irrelevant picture in the entire book. Many students emerge from an abstract algebra course with no sense of a 'group' as the central feature of 'symmetry'. In fact, 'symmetry under a group' is the very definition of 'a geometry' – a point I will return to below.

The recent literature in educational psychology and cognitive science confirms this broad cultural (mis)perception that geometry is marginal within mathematics. Outsiders automatically associate 'mathematics' with formulae, algebra and maybe analysis. A recent (and very interesting) book on mathematical cognition identifies 'numbers' and the abilities based on them (e.g. algebra) with mathematics [9]. When a scholar of multiple approaches to learning, such as Howard Gardner, considers 'mathematical intelligence', he identifies mathematics with a single approach involving logical sequences of formulae and sentences [20]. In his description, 'mathematics' is cut off from the "visual intelligence" and the "kinetic intelligence". When a book for teachers [2] describes the theory of multiple intelligences and the associated careers, the 'mathematician and scientist' are associated with the logical /mathematical intelligence, while the visual intelligence is associated with 'artist and architect'. Similarly, in this description the culture values 'scientific discoveries, mathematical theories, counting and classification' from the logical/mathematical intelligence and values 'artistic works, navigational systems, architectural designs, inventions' from the spatial intelligence. Of course these outcomes associated with the logical/mathematical intelligence are valuable. Unfortunately, geometry, as associated with the visual intelligence, is presented as marginal in mathematics and in science.

In short, many mathematicians present a public face to their students and to other intellectuals that mathematics (at least higher mathematics) is essentially about the logical intelligence [8].
The popular culture sees mathematics as detached from the spatial (visual) intelligence. From this point of view, the visual and the geometric are not an essential part of mathematics. Where 'geometry' appears it is quickly made analytic and treated as a source of calculations to illustrate the 'important areas' of math like algebra and calculus. This public face for mathematics is an important cultural result of the decline of geometry.

Does this decline matter? Does the public and educational disconnection of 'mathematics' from 'geometry' matter? Perhaps the current state of undergraduate and high school geometry accurately reflects the value of geometry to the learning and the future of students. Clearly the current curriculum is crowded and we have to cut to make room for important new mathematics. Is geometry now part of what someone called the 'saber tooth curriculum'? Have we now 'got it right' that geometry *should* decline?

II. Geometry is Rising.

Discrete geometry is already rising as an area of research inside and outside mathematics. Geometry is beginning to rise as an important area of learning and teaching. In this section I will focus on three distinct trends to support my assertion that geometry has more life now than two or three decades ago. In the following sections I will then offer some comments about what geometry is now and how we should teach it.

(A) Applications of geometry: new results and new problems. Geometry is again very active as a field of research in many disciplines and industries today. This work has generated new geometric problems, use new geometric results and even generate new areas of geometry. Sometimes this activity includes mathematicians and mathematics departments; often it is centered outside of 'mathematics'!

(B) Human abilities - visualization. Geometry is central to a basic human ability - visualization and reasoning with visual and spatial forms. For a variety of reasons, often associated with computers, this ability is playing an increasing role in learning, in memory, in communication, in problem solving and the practice of many professions.

(C) Suitable resources for learning. The development of dynamic geometry programs for teaching and for research is dramatically changing what researchers, students, and therefore teachers, do when they solve problems in geometry. Companion resources for teaching geometry in a rich way are accelerating the impact within the undergraduate and secondary classrooms of the rise in geometry at the level of research and applications.

(A) Applications of Geometry: new results and new problems.

Numerous current applications have a strong geometric component. In many cases, the problem includes getting 'geometric' information into a computer in a useful format, solving geometric problems, and outputting this solution as a visual or spatial form, as design to be built, as an action to be executed, or as an image to entertain. Solving these problems requires substantial geometric knowledge and people using the results of the research also benefit from a basic understanding of the geometry involved. Here, briefly, are a few illustrative examples.

(1) Computer Aided Design and Geometric Modeling.
A basic problem is to describe, design, modify, or manufacture the shapes we want: cars, planes, buildings, manufactured components, etc. using computers. The descriptions should be accurate enough to directly control the manufacturing and to permit simulation and testing of the objects, prior to making any physical models. For example, the most recent Boeing plane was entirely designed inside of the computer, without any physical models. Here are a few samples of the geometry involved in such work.

- Consider the hood of a new car. How is this described in the computer? We could input a bunch of points - but that does not give the surface, nor a 'picture' of the object during modifications of the design, nor instructions on how to manufacture the surface. Instead, the 'surface' is divided into regions and each region is described by some simple function that approximates, or even passes through, the initial points. These pieces have 'control points' to modify their shape that are combined to ensure that the pieces fit together in a geometric sense: continuity - no gaps in the hood, continuous derivatives - no sharp creases. Even continuous third derivatives are needed for display in the showroom where our eyes can 'see' such details.

The standard mathematical objects are called ‘splines’ and they are a striking generalization of those strange problems in calculus in which we piece together parts of two polynomial functions to make a single continuous function (Figure 3) [22].

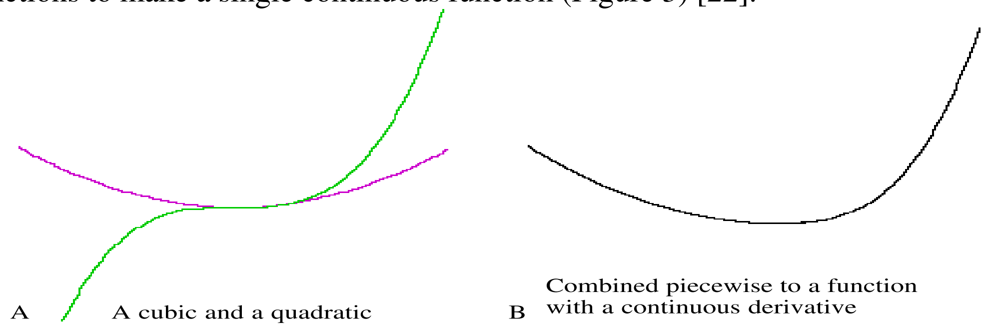

Figure 3

- Constrained CAD. Given a structural design or the ‘measurements’ of an object, how do these measurements determine the shape? Which measurements can be changed, without altering other measurements or constraints?

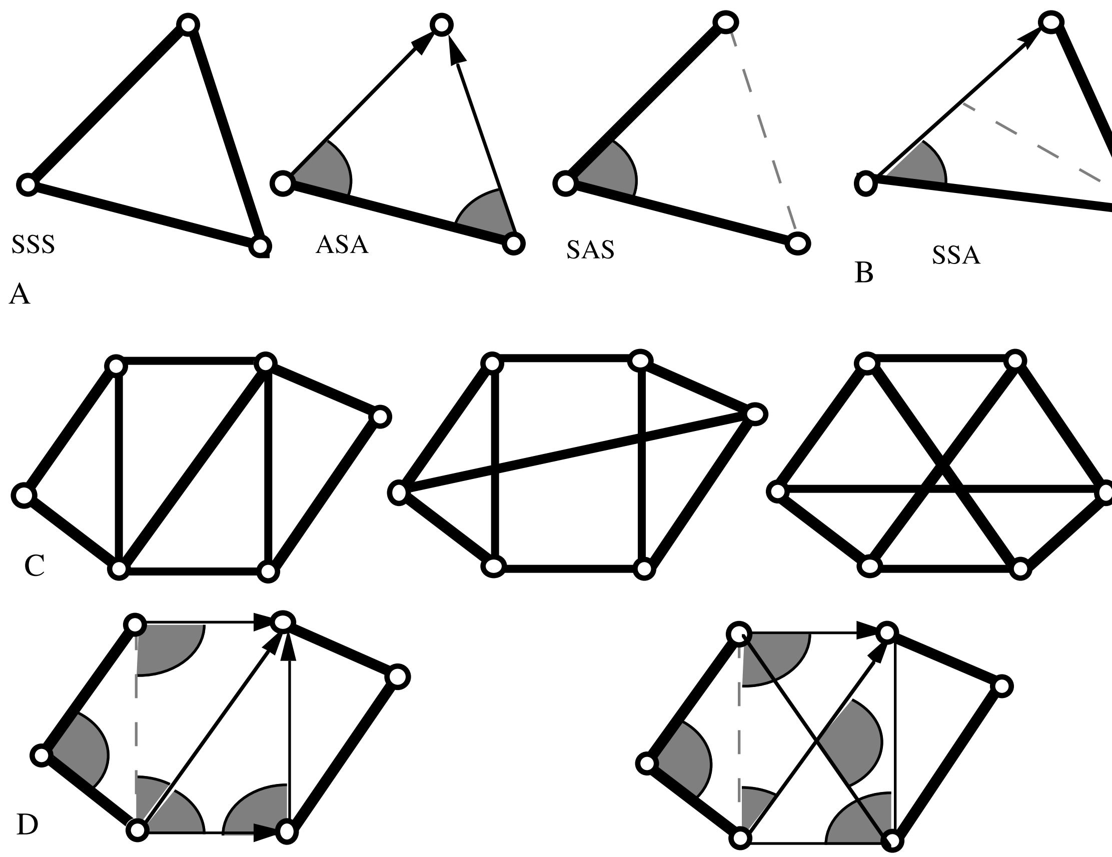

Figure 4

- Consider three points in the plane Figure 4A, B. We can choose three measurements (angles, lengths) then the others are determined. That is an important part of the content of the congruence theorems SSS, SAS, AAS (Figure 4A). What constraints will make these unique up to congruence, or at least up to congruence in a neighborhood as with SSA (Figure 4B)? What about six points in the plane? Figure 4C,D illustrates some patterns that guarantee local uniqueness, up to congruence. What about 1000 objects in the plane - and algorithms for handling these? (The basic count for such independent constraints in the plane on n points is 2n-3, see [22, 59].) What about the analogous theorems and questions on the sphere?

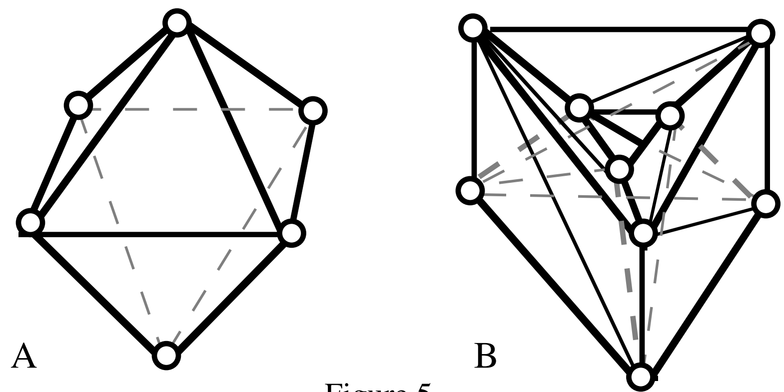

Figure 5

- Consider a triangulated sphere in 3-space (Figure 5A). All convex triangulated spheres are rigid (locally unique up to congruence) in 3-space, as illustrated by domes and natural structures. All of the edges are independent constraints and we can change in one a small amount and the structure will not crack [58]. All of this is not the observation of Fuller but of Cauchy and is a property embodied many naturally growing organisms. How about other patterns and other objects? What about the triangulated torus (the types of objects projected for space stations etc.)? That is a subject of current research (Figure 5B). In general, we ask: can the computer also generate the overall change in the structure from the small changes in the lengths?

How these kinds of questions are answered (exact symbolic expressions, numerical approximation, etc.) is a basic issue determining the 'competitiveness' of CAD engines which run inside AutoCAD or the programs at Boeing, GM, Mercedes, etc. [44]. The people working on this are generating numerous new geometric problems and are using everything we know about straightedge and ruler constructions, etc. Of course, the designer is not willing to wait for days to see the results of a change - so the responses must be efficient, often real-time.

(2) Robotics.
To use a robot, we must input (cameras, sensors, prepackaged information) a geometric model of the environment. The whole issue of what vocabulary will be used (e.g. solid modeling – Boolean combinations of basic shapes, polyhedral approximations etc.), and how the information will be structured is a major area of research in a field called 'computational geometry' [43].

With this model in the computer, we must plan what can be moved, and along which paths (motion planning). Can we get the object from here to there with the robot? Using this 'motion plan', we must determine which sequence of actions by the manipulators and the locomotion will move the objects along the planned path (inverse kinematics). At every step, there are geometric problems to be solved - and they must be solved efficiently.

These problems alone have generated books of new results, new forms of old geometric techniques, and new questions [22]. There may be more graduate computer science courses in computational geometry than graduate mathematics courses in discrete geometry today.

(3) Medical Imaging.
We want to use non-intrusive measurements (pictures) to construct an adequate three-dimensional image of parts of the body. For example, a series of projections or images from ultra-sound, or MRI from several directions or points are collected. How many measurements do we need to construct the full three dimensional image? What algorithm can we use to reconstruct the full image from the pieces? Again, lots of geometric problems, lots of research and some substantial new results in fields like geometric tomography [21].

(4) Computer Animation and Visual Presentations.
How can the computer generate sufficiently rich images to fool our human perceptions of the static form and the moving objects? Experimental movies such as ‘Gerri’s Story’ are exercises in substantial mathematics with a clear geometric component. The current version of the video ‘A Bug’s Life’ contains this Academy Award winning short. One of the computer scientists/geometers who worked on this film described it as exercise in handling texture and modeling clothing with new levels of mathematics. New mathematics with a geometric base, such as fractals, are a piece of this work. So is geometric modeling: one of these key developers at Pixar moved from academic research working with Boeing on geometric modeling.

(5) Linear Programming.
In business, a widely used ‘new’ piece of mathematics for scheduling and for decision making is linear programming - the optimization of some function (e.g. low cost or high profit) under constraints. The basic conceptual processes for linear programming, and a number of the innovations in linear programming, have a substantial geometric basis (polyhedra, higher dimensional polytopes, and duality). One of the big puzzles in this area is why the algorithm is so efficient - and how to predict the special situations where it will not be efficient. Sometimes the algorithms themselves are ‘dumbed down’ because a correct understanding requires familiarity with projective geometry and few people have that familiarity these days.

There are comparable or related geometric problems arising in chemistry (computational chemistry and the shapes of molecules), material physics (modeling glasses and aggregate materials), biology (modeling of proteins, ‘docking’ of drugs on other molecules), Geographic Information Systems (GIS), and most fields of engineering. As a reflective chemist said in a recent lecture: ‘chemistry is geometry’. Some sources of information on these developments would include [3, 22, 43, 52].

In summary, geometry is out there and it is essential for application. Geometry will be practiced, with or without mathematicians, and with or without an education in ‘geometry’. I believe this geometry would be done better if the future practitioners of geometry receive an appropriate preparation in geometry. The ‘geometry gap’ will haunt North America.

(B) A Human Abilities – Visualization

Many sources confirm that human intelligence (collecting information, organizing and remembering it, reasoning and problem solving with it, communicating it) is a mix of many distinct interconnected abilities [20]. One package of these abilities, developed through our visual perceptions and our visual experiences of the 3-D world, augmented by our kinesthetic experiences and intelligence, I will call ‘visualization’ for short. Let me cite some examples and evidence.

- From our earliest months, our visual apparatus is one of our richest sources of stimulation and information. Vision and spatial perception is richly wired into the brain, with amazing capacities to process and interpret [28, 35]. Recent studies of our ‘visual intelligence’ in the sense of direct perception already demonstrate in a dramatic fashion that we construct what we perceive. I strongly recommend the recent book by Donald Huffman: Visual Intelligence: How We Create What We See [28]. A fascinating part of this analysis is the rich set of skills, with their deep, implicit mathematics (geometry and topology) which the child develops by about age one and continues to develop throughout life. By age two, given a pattern of shifting features, we ‘create’ a single rigidly moving 3-D image if that can fit the perceptual data [28]. I encourage you to check the associated web site for some illustrations of what we create! Visual work in general, and spatial (geometric) work in particular, build on one of our richest sources of information and highly developed set of cognitive abilities.
- Today, when people want to display rich sets of data in statistics or other sciences, seeking ‘patterns’ to understand the information, they use rich, carefully designed images. There are

substantial efforts to encourage people to ‘visualize’ data for work in statistics and other related fields. The recent book Visual Revelations wonderfully illustrates the value of putting together pieces of information in overlapping visual pattern [56]. The books [53, 54] are other classic collections of information and data in visual form. Books on ‘Scientific Visualization’ are spreading the images of what is now possible and desirable in many fields of science [11].

- We speak of ‘imagining’ (imaging) ideas and experiences in our heads. Recent studies in neurology and cognitive science confirm that this internal imaging uses brain processes and ‘spatial’ search techniques in common with what we do when we ‘inspect’ an external diagram or object [35]. The two experiences can be considered as parts of a single whole.
- In problem solving, images and diagrams play an effective role in (re)organizing the information into associated parts and even coherent wholes in a ‘gestalt’ that are very different from how we work with language or formulae [37, 41].
- Much of our ability to use the objects and devices which augment our memory and our ability to control our environment (things that make us smart [42]) depends on good visual design of interfaces to indicate, without words, what can be done and what effect our actions are having.
- With good notation, steps in algebra are determined by ‘appearance’ in essential ways. What I do in the next step in a problem is based on what I ‘see’ and how the current step is presented. Much effort in algebra is spent changing appearances to evoke the correct next step. Of course, done correctly, these are controlled steps. As I tell my students, algebra is cosmetics, not surgery: change the appearance but not the substance.

There is a rich interdisciplinary field of research under headings such as ‘diagrammatic reasoning’, ‘thinking with diagrams’ and ‘spatial reasoning’. (See for example [15, 37, 38]). These studies bring together work on cognitive science, artificial intelligence, design, history of science, pedagogy, human-computer interfaces, philosophy, and creativity, among others. There is now a rich literature about the role of diagrams and geometric reasoning in effective learning and creation in a variety of professions, including mathematics. Much less is known about how individuals actually use diagrams and about the roots of the substantial differences in ability among individuals.

Moreover we can change the way we ‘see’ things. I have recently been working to develop a course on visualization, as my report elsewhere in this volume describes [60]. I have been struck by the consistent messages from books such as ‘Drawing on the Right Side of the Brain’, Thinking Visually, and ‘The VizAbility Handbook’ [17, 38, 61]. Their goal is to ‘change the way you see’ and this can be done. This is an important message that I will return to below - most people can improve their visualization and spatial reasoning.

When we compare different individuals on their visualization skills, we find substantial variation. For example, tests confirm that people have a wide variety of skills with mental transformations, and with related problem solving skills involving spatial and visual reasoning [63]. The role of such transformations (learned ways of mentally modifying diagrams) and how these connect with specific skills developed from geometric experience and for solving geometric problems would be an important area to understand. Much of this research is currently being done by non-geometers.

It is tempting to assume that visualization, while valuable for some people, is not essential. However, for some people visualization is their essential mode of reason. Consider work by and about high functioning autistics which indicates that some of them are essentially visual learners. See, for example, the autobiographic: ‘Thinking in Pictures: my life as an autistic’, by Temple Grandin, a tenured university professor who designs facilities for handling domestic animals [23, 50]. There are people who make very little use of visual and geometric reasoning and there are people who rely on visual reasoning almost entirely.

What is the role of visualization in mathematics? Tommy Dreyfus [16] gives a good survey of related literature as well as the impact of computers on visualization. Everything we are finding out about the how the brain works, about the variety of people's 'intelligences', and about the different parts of the brain that are active for different approaches to a 'mathematical' problem, confirms how distinct our approaches to mathematics can be and how rich it is to combine multiple approaches. As an example, I mention something from the Science last April [10]. Experiments with bilingual people, and two types of problems confirmed that explicit numerical calculations involved the linguistic centers, but approximate estimations about the same numbers involved other parts of the brain (parts closer to the control of the fingers and perhaps to spatial reasoning) [9, 10]. The role of geometric and spatial cognition in how our brains 'do' mathematics (mathematical cognition) is a rich area for research and insight.

More often than most mathematicians admit in their public communication, creative mathematics is done in visual forms. For some classical descriptions of this, I recommend the book of Hadamard [24], in which this role of the visual in creative mathematics is a central theme. Moreover, many parts of mathematics have a geometric counterpart - a counterpart that may give an important sense of 'understanding' and 'insight' [8]. However, the public culture of mathematics has downplayed the role of the visual - as at best an analogy and at worst an inadequate substitute for the 'real mathematics' of theorems and proofs (done in formulae and language). Here is a quote from the mathematician J.E. Littlewood [8 p.xi]:

> "A heavy warning used to be given that pictures are not rigorous; this has never had its bluff called and has permanently frightened its victims into playing for safety."

Mathematics has many faces and needs people with many different approaches.

When I have spoken with reflective high school teachers about why geometry should be taught in high school, they often respond in terms like: "I need geometry and the activities it opens up in order to involve some of my students in mathematics. It is delightful to see how roles change and students who were struggling are now the leaders in certain activities." Mathematics needs to be taught in an inclusive way that helps the visually strong people to connect to the core of mathematics and also see themselves as empowered users and creators of mathematics. Geometry can play an important role in inclusive teaching of mathematics.

A key point is that many of these outcomes such as scientific discoveries, new mathematical theories, and problem solving in general have substantial visual/spatial components . In some patterns of dyslexia, visual strengths more than compensate for weakness in reading and writing words [57]. Both of these skills, and more, should be presented as core abilities to bring to science. Listening to my class on visualization for first year science students, it is clear that all subjects could do a better job of integrating and teaching the use of visual in science. It is also clear that, in general, mathematics classes do a particularly bad job of this.

In short, visualization is a rich ability for many people and can play an important role in fields. All subjects, including mathematics, should aggressively incorporate this ability into their public and private practices. My regret is that we have developed a curriculum which convinces many people who are 'algebraic' that they do math well and do not need other skills, and convinces many people who are 'visual' (geometric) that they do not belong in mathematics. I will return to that below.

So far, these observations lead me to two connected conclusions for students: the value of visualization in a variety of careers and professions; and the value and possibility of increasing their abilities in visualization and spatial reasoning. Geometry has a role to play in both of these.

(C) Suitable Resources for Learning.

In my talk, I had the liberty to include overheads from books and articles, some animated images from my computer, and even some objects to be seen and manipulated. I have described the great value of appropriate use of geometry and visuals. Why is there so little 'visual' in this paper and in mathematical presentations in general?

When I was drawing up this paper, I had still and moving images for almost every paragraph. The difficulties I faced represent some the restrictions of print as a medium and of my resources:

- the lack of simple tools for the preparation and display of some of these images;
- the conventions of copyright and 'fair use' for images compared to conventions for text;
- the fact that many visual conventions I have for myself are private – not part of the shared conventions of mathematics (this is a vicious cycle).

All these mean that I am doing something very perverse in this section: using text to describe the impact of visuals, rather than the visuals themselves. My students have reported similar experiences. Even when CDROMS are used in class and in studying, they face exams that are almost entirely text – and have their own expectation that answers will be essentially text. They have not learned (or been taught) the effective use of visuals for their communication. My very difficulties illustrate, in part, our standard text rich, visual poor presentations of mathematics including geometry.

The last decade has seen an increasing number of dynamic geometry programs for teaching and for research. Some of these programs, such as The Geometer's SketchPad and Cabri, were originally designed for teaching geometry at the secondary level, but they quickly became tools for researchers as well [6, 31, 33].

People who wish to solve plane geometry problems, at any level, find these animated tools for explorations and investigation are invaluable in adding precision and dynamic transformations to what would otherwise be a static, possibly crude, external representation. Although we start with an unlimited supply of 'planes' (pieces of paper) to experiment with, we find that these new tools make us 'smarter' in geometry [42]. We (teachers and students) can now conjecture, generate a rich set of examples or counter-examples, and extract visual patterns and processes for reworking into proofs of various types, including visual proofs. The recent article of Philip Davis [14] highlights how dynamic geometry programs and other computer aids have transfigured the study of triangle geometry. Anyone reading recent exchanges on the geometry lists at the Math Forum [39] will have observed the depth, the passion, and the insights that are being reported from classrooms and researchers across the world.

A new generation of tools designed for geometry research, such as Cinderella [48], are spreading this impact from the classroom and plane Euclidean geometry to the classical geometries (Euclidean, hyperbolic and elliptical), with multiple models and broader transformations. At best, we start with a very limited supply of physical spheres or 3-D pieces and objects [51]. These new programs, and similar programs for polyhedra [45, 47], bring the playfulness and the precision of plane dynamic geometry to these other fields of geometry.

These tools for transforming the practice of geometry are spreading. However, effective programs for the full range of 3-D geometry are still not available. Given an algebraic formula, we can generate 3-D displays in Maple or Mathematica. We can even use a mouse to change the view of this displayed object. However, this remains a long way from our experience with object in the world – and the spatial cognition that goes with that experience. The struggle centers around input devices that capture what our hands do in space, and on control over displays which capture what our head and eye movements do with real objects. I anticipate that some of these difficulties will be solved in the near future.

As is richly illustrated by the recent book [33], these dynamic geometry programs can change what questions we ask and what methods we are likely to use. Increasingly, these programs permit easy display of dynamic images on the web, in machine independent Java. I know that when I do visual and diagrammatic reasoning, I often run 'animated movies' in my head. Such images are now accessible for communication with our students. They are also accessible for the students to use, without years of hit and miss learning or fumbling with inaccurate ruler and compass constructions where 'concurrent lines' never seem to meet. Moving these images from an internal image to an external form has a dramatic impact on the role they play:

- the precision and reliability is increased, particularly for the beginner;
- the range of examples experienced in a short time changes by an order of magnitude;
- students experience 'invariance' of properties over changing examples (a fundamental concept of geometry – see the next section);
- students move to a higher level of analysis and synthesis, as illustrated by the difference between a 'drawing' and a 'construction' in these programs.
- we establish both shared experiences and common conventions for the classroom community and the larger community;
- we have improved communication among the users, based on these common experiences;
- we even have more closely shared internal images - extracted from the shared external forms.

What we can do in the geometry classroom or laboratory has changed. This generates a dramatic shift in how geometry is practiced and in what we 'see' (and who sees it).

Changing our tools generates critical changes in our subject. The history of science offers an insight into the impact of a change in technology. In the history of science, 'instruments' play a critical role in the development of any experimental approach. Dynamic geometry, complete with measurements, constructions etc. plays the role of our new 'instruments' - replacing the earlier (and more limited) compass and straightedge instruments, or the transitional 'instrument' of origami and paper folding. We now have the basis for 'experimental geometry' at a very high level. While these programs primarily produce images (often moving images), they can also produce numbers, and tables of numbers (graphs). In this, they very much resemble instruments in physics or biology - extending our senses and our kinetic experiences to a level that becomes a qualitative change. As in experimental sciences, these instruments raise the possibility of 'moral certainty' that something is true, even though we do not have a mathematical proof (and therefore may be unclear about the exact assumptions being made) [25 pp.229-231]. This in itself is an interesting debate [8].

Companion resources, which embody active collaborative learning with open-ended problems, a rich variety of connections, and strongly visual presentations, including the appropriate use of manipulatives, have been developed for teaching geometry [27]. Dynamic geometry programs become one of the tools for collaborative learning. While these resources assume a limited class size and an engaged student body, the experience among those using the resources is that the students quickly become engaged, provided the other resources of space and limited class size are provided. The only catch is that these active collaborative methods are not common in other mathematics classrooms and some students find this transition hard. For the same reason, some of the graduates of these courses find that they cannot sustain these methods in their own secondary classrooms, under the pressure of a crowded curriculum and the lack of resources and support.

Effective changes in the geometry classroom depend on changes in how other parts of mathematics are taught, Changes in how all of math is taught becomes an associated goal. There are new programs to teach statistics (at the high school level) in a highly visual way, coming from the same people who developed dynamic geometry programs [32]. There are programs to 'visualize' what is happening in differential equations and dynamical systems. Their visual form and appeal drive the interest in fractals, for teaching and for many applications. Fractals have substantial applications in graphics for 'natural objects'. Fractals are also being used for image compression.

The study of fractals and dynamical systems would not have evolved as rapidly without our powerful tools for visualization. Other fields are experimenting with the appropriate combination of diagrammatic and algebraic reasoning [64]. These, however, are the topic for another occasion.

There are now materials for a variety of geometry courses for non-mathematicians who are called on to use 'geometry' in their own fields. At a school such a Cornell, there are about eight different undergraduate geometry courses and the enrollment is strong. There are also courses taught in other departments, such as computer science, which contain substantial portions of geometry. As with many other parts of mathematics, if the need is there and the mathematics department does not meet that need and meet it well, parts of the teaching of mathematics will migrate into the other programs. That will be our loss and perhaps also a loss for our students.

Must there be a long 'lag' between the revival of geometry at the research level, in applications and in mathematical work, and the revival of geometry in the high school and elementary curriculum? It is possible that these new resources and our awareness of the trends to increasing use of geometry and visualization can shorten this lag.

This will only happen if these tools and this vision are in the hands and the minds of the next generation of teachers. The recent province wide purchase of the Geometers Sketchpad for all schools and teachers in Ontario, and the companion requirement to use dynamic geometry from grade 9 on in the new curriculum, are encouraging signs. If supported effectively by Faculties of Education and Departments of Mathematics in our teaching and our approach, we can accelerate the shift towards geometry and the visual.

III. What is a Geometry?

Having stated my case for a revival of geometry, I should be more precise about what geometry is, and some conclusions we can draw for the curriculum and the pedagogy of geometry.

The 'modern' definition of geometry, due to Felix Klein in 1870, states that a geometry is a space with a group of transformations into itself [34]. The geometer studies the properties which are invariant (unchanged) under these transformations. There are different 'geometries' connected by different groups of transformations, forming a hierarchy of geometries.

Consider the visual presentation of the hierarchy in Figure 6. This shows (via inclusion) the groups of transformations increasing from the restricted congruences of Euclidean geometry, through the shearing and parallel projections of affine geometry and the central projections of projective geometry to the general continuous maps of topology. As the group of allowed transformations gets larger, fewer properties are unchanged (invariant). The concepts we will study get fewer, the vocabulary gets simpler, and the theorems get more general. More and more objects are 'the same' up to the 'symmetries' induced by these transformations.

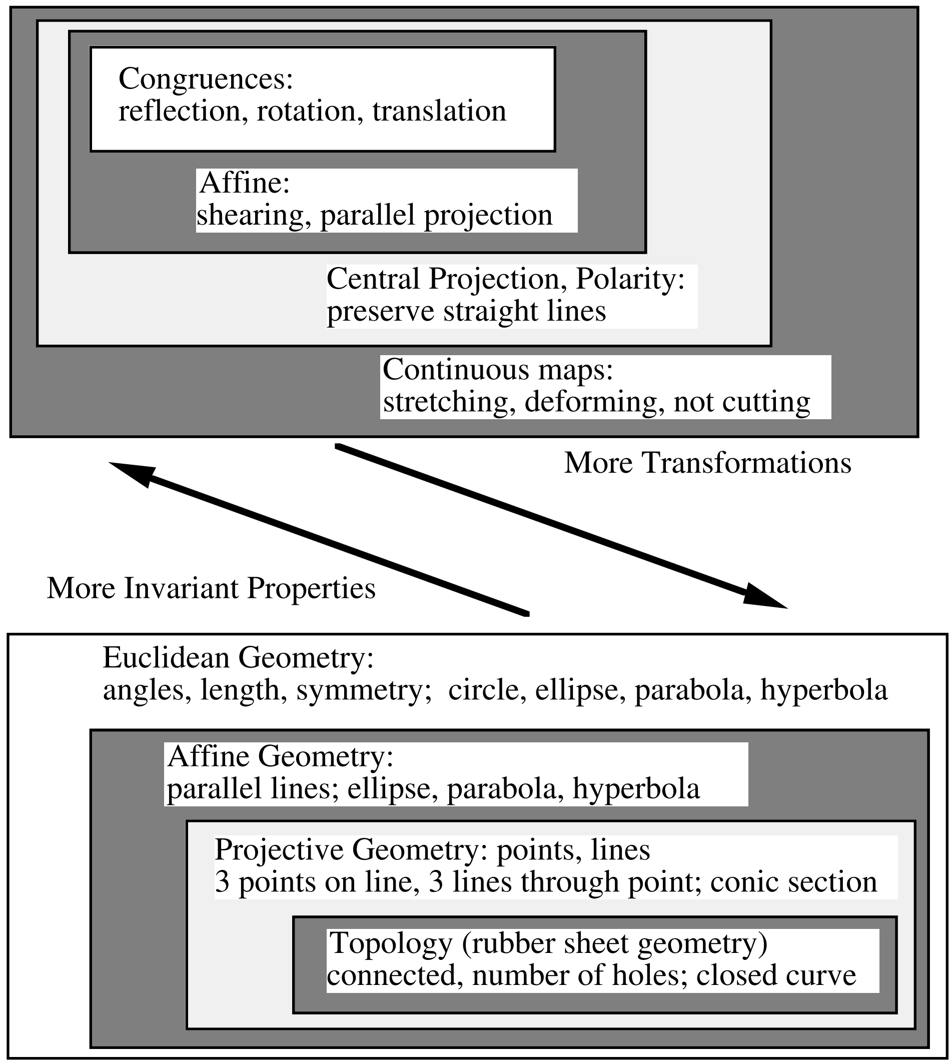

Figure 6: Klein's hierarchy of plane geometries

For example:

1. In topology, all simple polygons (even all simple closed curves) are 'the same'. For any pair of simple closed curves in the plane, there is a reversible continuous map (a topological homeomorphism) which takes one onto the other. A typical topological theorem is the Jordan Curve Theorem (Figure 7): a simple closed curve has an inside and an outside.

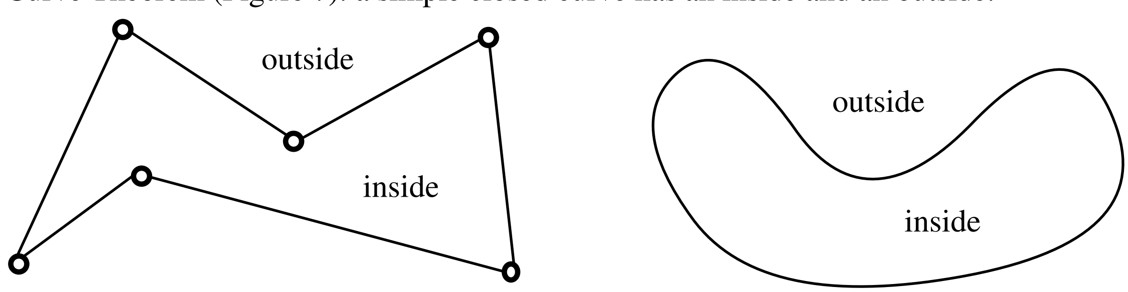

Figure 7: A topological theorem

2. In projective geometry, all quadrilaterals (except those with three points collinear) are 'the same'. That is, given any two quadrilaterals, with no three points collinear, there is a projective transformation (a collineation) which takes one onto the other. Straight lines, points of intersection, and associated properties (such as six points sharing a conic) are preserved. A sample theorem would be Pascal's Theorem (Figure 8) that a hexagon on any conic section has alternate pairs of sides meeting in collinear points. Once you prove this for a circle, projective transformations will prove it for any (non-degenerate) conic section.

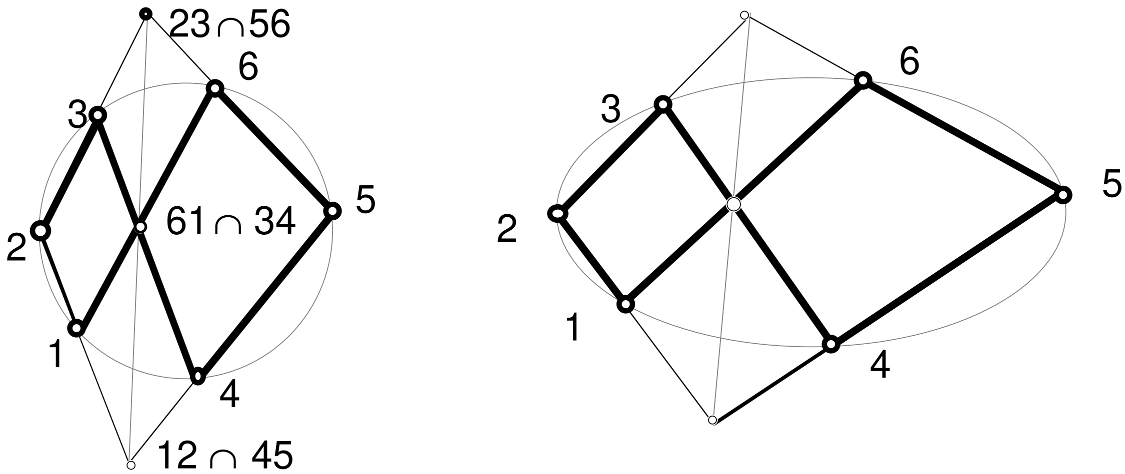

Figure 8: A projective theorem

3. In affine geometry, all non-collinear triangles are 'the same'. That is any non-collinear triangle can be taken onto any other such triangle: the equilateral triangle is the 'typical triangle'! A sample theorem would be that the medians of a triangle meet in a point (Figure 9). Since this is true for the equilateral triangle, and medians are preserved by affine transformations, we have a simple proof for all triangles. All parallelograms are the same, so the square is a typical parallelogram. Thus another typical theorem would be that the diagonals of a parallelogram bisect one another.

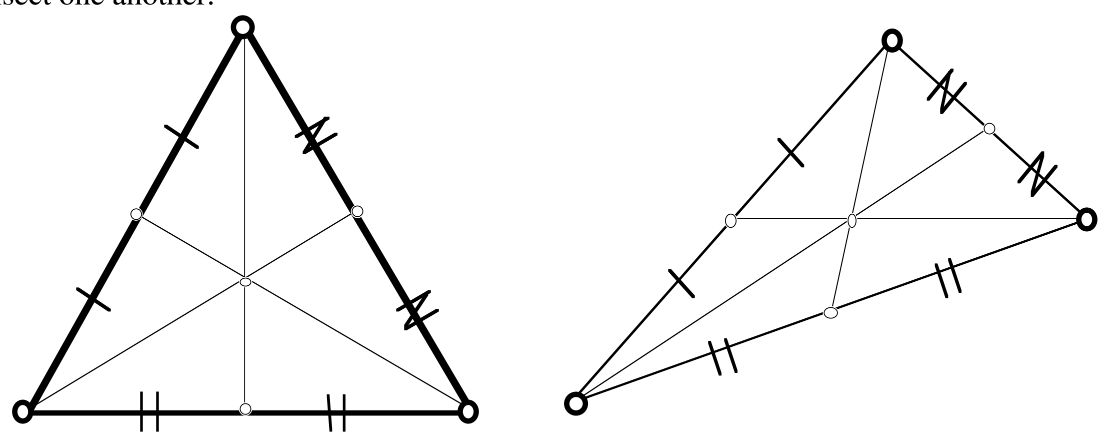

Figure 9: An affine theorem

4. Finally, in Euclidean geometry we distinguish triangles by the size of their edges and the size of their angles. A typical theorem (nicely proven by reflections) is that the right bisectors of the sides intersect in a point (the circumcenter) (Figure 10).

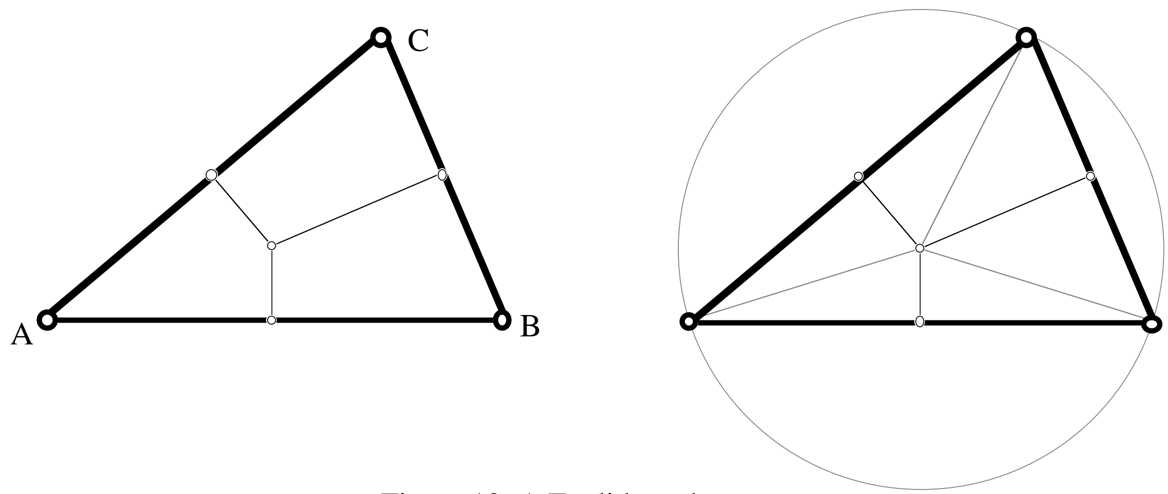

Figure 10: A Euclidean theorem.

There are several practical points that emerge from this hierarchy of geometries:

(i) 'Transformations' are the key concept of geometry. Reasoning with transformations should be a central theme of our learning of geometry [62]. Patterns, the very core of mathematics, are about invariance or sameness under certain transformations.

(ii) The more transformations you have, the more objects are 'the same' and the simpler the properties and vocabulary will be. If you can use topology or projective geometry rather than Euclidean geometry, the thinking and writing will be simpler. Knowing only Euclidean geometry is like having only one tool – a hammer. Everything becomes 'a nail to be hit' and many tasks cannot be done effectively. Geometers need to know, to 'see' the patterns of many classes of transformations.

(iii) There are many groups and many geometries - and adept people will choose which geometry to use for a particular problem with care. For new geometric problems, the first crucial task to is to decide: which geometry? Unless the problem is correctly placed within the geometric hierarchy, there is a substantial risk of:

- either burying the pattern in a mass of irrelevant detail by being too low in the hierarchy so that little effective can be done;
- or loosing the pattern completely by being too high in the hierarchy.

On several occasions in my work in applied geometry, I have found people stuck 'too low' in the hierarchy, lost in a maze of irrelevant details and unaware of the level of invariance and the powerful tools that a better, 'higher' geometry would bring [59]. On the other hand, as Einstein said, "things should be a simple as possible and no simpler". I have also encountered people working too high in the hierarchy, where nothing correct can be worked out because the properties studied are not invariant under all the transformations. The quality of the answer to this fundamental problem of 'Which Geometry?' will shape the entire study.

(iv) Plane geometry, spherical geometry and other geometries can be studied, compared and connected to see different and comparable forms of geometry [27]. This too is part of the geometric hierarchy. For example, many of the common theorems of Euclidean, spherical, and hyperbolic geometry lie in the shared projective geometry that lies above them all in the larger hierarchy. This common projective geometry is also important for application [22].

(v) People learn basic skills in various geometries at different stages of development. There are a number of indications that, roughly, children develop 'down' the hierarchy from topology (first) to Euclidean geometry last, as was indicated by Piaget [30, 63]. Certainly by age four, children know in a practical sense, what is connected and what is not (what they can reach, or whether a new space includes a 'race track' they can run around and return to their starting point).

(vi) Children learn 3-D transformations before they learn 2-D transformations [28] and they learn the 3-D geometric hierarchy before they learn the 2-D hierarchy [4 p.6]. Experiments in the former Soviet Union on spatial reasoning indicated that even elementary students have substantial abilities with projections and 2-D representations of 3-D objects, and can learn more than is usually taught, if we think it is important [63].

(vii) Our teaching of geometry, from the plane up, disguises and even blocks that knowledge [63 p.200]. In short, we teach geometry in the reverse order of children's development, from the bottom of the hierarchy up and from 2-D to 3-D, through most of the K-10 curriculum.

As you will see in the next section, an understanding that geometry is about transformations is central to my view of how geometry should be taught.

IV. Reflections on Teaching Geometry into the Next Century

Which would you choose? A geometry class is like:
(a) a trip to the dentist or the doctor;
(b) a trip to your favorite restaurant.
From the student comments I see on the internet, the common answer would be (a). However, from what I also see on the internet, with a different environment, geometry is an area of mathematics that is highly engaging and can generate high quality learning for a wide variety of people. A basic issue is how geometry is taught.

I offer a brief summary of some conclusions about the teaching of geometry that I draw from the previous discussion.

1. The overall curriculum should teach geometry 'down the hierarchy' - from topology, through projections and finally to Euclidean geometry, making visible the connections that the students themselves have already learned.

We ignore the rich talents of young students at the upper levels of the geometric hierarchy and disconnect from this experience at our peril.

2. Start 'geometry' early - students can do a lot more than we credit them with.

At a very early age, children develop a very rich 'visual intelligence' in terms of perception and experiences. They have questions and lots of these questions and explorations can be connected to geometry if we use the right types of physical and visual presentations. They have developed advanced visual skills for which the precise vocabulary is 3-D differential geometry and differential topology [28, 36]. I would not propose we use this vocabulary but I would propose we do not ignore, even suppress these visual abilities. Instead we should connect with these abilities.

3. Teach visualization, transformations and spatial reasoning using manipulatives and graphic representations in a systematic way. An important and achievable goal is to expand the way that students 'see'.

We often don't teach the use of visual tools and visual reasoning in any systematic way [1]. We just test it as something obvious – 'the students will see what I see' - or else we avoid the use of visual tools because we (correctly) predict that a substantial number of students will not 'get it'.

The van Hiele model of geometry education reminds us of the basic pedagogical lesson that students learn from their experience up and we cannot lightly skip over basic stages and basic connections [29]. Even at University today, we cannot assume that the basic levels of experience, vocabulary and communication are in place. This imperative that we work through all the van Hiele levels in any new geometry, including the use of appropriate software and manipulatives, applies to the undergraduate geometry classroom. It is important both for the learning of the students, and as a model of how other geometry can be learned and should be taught in their future classrooms.

4. Teach transformations with animated images and not just static images.

The tools are now available to bring this into the classroom and to bring these home, on the web or the home computer. Transformations and change within geometry are central to understanding geometry, as we experience it [36] and as we apply it (see point 9 below).

5. Use 3-D from the beginning, along with representations and transformations.

Children and adults live and see in three dimensions [49]. Neglecting this in early geometry education actually interferes with appropriate transfer from this experience into working with geometries. To quote from the NCTM standards:

> "In grades 5-8, the mathematics curriculum should study the geometry of one, two, and three dimensions…so that students can visualize and represent geometric figures with special attention to developing spatial sense."

3-D is our primary experience and 2-D representations are highly ‘conventional’ and difficult to ‘read’. We need a rich curriculum on representing, analyzing, and constructing 2-D images for 3-D objects and 3-D objects from 2-D images [1, 4, 40]. However, I would start earlier than grade 5.

6. Include visual forms throughout the mathematics, at all levels, to include more of the students within the tent of ‘those who are good at math’ and to enrich the range of approaches of all students.

Mathematics needs to be taught in an inclusive way that helps the visually strong people to connect to the core of mathematics and also see themselves as empowered users and creators of mathematics in the sense of geometry and information that can be encoded in geometric forms.

7. Include applications of geometry and integrate consistent visual forms throughout the curriculum [12].

Visualization and the use of geometric forms and geometric reasoning runs across many subjects: geography, science, technology, and computer science. Just as it is important to use consistent notation for numbers and words, it is valuable to use consistent visual forms in a variety of situations and years. Without this, there will be very limited visual communication.

8. Geometry should return as a central part of the undergraduate curriculum, both for future mathematicians and as a service course for a variety of fields.

Geometry has an important role to play for mathematics majors and for future professionals in may fields. Every undergraduate program should include appropriate geometry courses aimed at these varied groups. Such courses should be rich in the use of dynamic geometry, manipulatives, transformations, and connections with applications. Universities that have many geometry courses find that enrolment is strong - there is the student interest.

9. Teach the geometric hierarchy within the core geometry curriculum.

Although the geometric hierarchy is not central to most current teaching of geometry, it is already central to working with geometric applications where invariance under transformations is a key issue. Any geometry course taught to people who will apply geometry must include an effective introduction to deciding where a problem lies in the geometric hierarchy. Any course taught to future geometry teachers should also include a solid working introduction to using the hierarchy.

10. An appropriate geometry course is an essential part of the training of future teachers.

All teachers of mathematics at the secondary and primary levels need to be comfortable with visualization and exploration in geometry. They need to be comfortable with multiple approaches to geometry, including dynamic geometry programs, and thinking with diagrams. In order to practice these approaches in their own classrooms, they must experience undergraduate courses that are rich in the use of dynamic geometry, manipulatives, transformations, and connections with applications. Teachers who have been cut off from the sources and practice of geometry cannot nourish the imagination and make the connections to the rich experience of the students.

This is something I have quietly believed for some time. Writing this article has encouraged me to be public with this proposal. I know of no Ontario Faculty of Education that makes geometry part of the required background even for a mathematics specialist, let alone someone with mathematics as a teaching subject. Like many of these ‘requirements’, the key may be the availability of ‘appropriate’ geometry courses. That is a responsibility of mathematics departments.

Geometry is alive. Let’s join in a celebration and in a conversation about how we support and enjoy this lively part of mathematics.

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