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Solution to the OK Corral Model via Decoupling of Friedman's Urn

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We consider the OK Corral model formulated by Williams and McIlroy[11] and later studied by Kingman.[7] In this paper we refine some of Kingman's results, by showing the connection between this model and Friedman's urn, and using Rubin's construction to decouple the urn. Also we obtain the exact expression for the probability of survival of exactly S gunmen given an initially fair configuration.

KEY WORDS: OK Corral; urn models; coupling; reinforced random walks.

1. INTRODUCTION

In this paper we extend the results of Kingman[7] for the OK Corral model using decoupling of an urn model into two independent continuous-time birth processes. Not only do we demonstrate an alternative and relatively simple method, but also we refine some of Kingman's theorems. On the side, we also obtain results on the speed of convergence for Friedman's urn.

The OK Corral process (X_t, Y_t), $t = 0, 1, 2, \ldots$, is a $\mathbf{Z}^2$ valued process used to model the famous gunfight. Its transition probabilities satisfy

$$\mathsf{P}((X_{t+1}, Y_{t+1}) = (X_t - 1, Y_t) \mid (X_t, Y_t)) = \frac{Y_t}{X_t + Y_t},$$

$$\mathsf{P}((X_{t+1}, Y_{t+1}) = (X_t, Y_t - 1) \mid (X_t, Y_t)) = \frac{X_t}{X_t + Y_t}.$$

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The process starts with $X_0 = Y_0 = N$,[3] and runs till either of X_t or Y_t becomes zero. The value of the other process at this time is denoted by S_N, further abbreviated as S. Williams and McIlroy[11] have found the correct scaling for the limiting distribution of S, which surprisingly turns out to be $N^{3/4}$. In Kingman[7] the limiting distribution of $S/N^{3/4}$ is found. The results were obtained using martingale techniques to compute the asymptotic moments of the above distribution. In our paper we obtain the asymptotic expression for the probability that $S = s$ for *each* s, provided $s = O(N^{3/4})$.

A seemingly unrelated model is an urn model introduced by Friedman.[5] In this model, the urn contains black and white balls. A ball is chosen at random, and then it is replaced by a balls of the same color and b balls of the opposite color. A special case of Friedman's urn (namely, $a = 0$, $b = 1$),[4] is the following process: (X'_t, Y'_t), $t = 0, 1, 2, \ldots$,

$$\begin{aligned} \mathsf{P}((X'_{t+1}, Y'_{t+1}) = (X'_t + 1, Y'_t) \mid (X'_t, Y'_t)) &= \frac{Y'_t}{X'_t + Y'_t}, \\ \mathsf{P}((X'_{t+1}, Y'_{t+1}) = (X'_t, Y'_t + 1) \mid (X'_t, Y'_t)) &= \frac{X'_t}{X'_t + Y'_t}. \end{aligned} \tag{1}$$

Freedman's[4] results for this particular urn give

$$\frac{X'_t - Y'_t}{t} \xrightarrow{\mathscr{D}} \mathscr{N}\left(0, \frac{1}{3}\right),$$

but do not say much about how fast (X'_t, Y'_t) would approach the state $X'_t \approx Y'_t$ if the process is started from an off-equilibrium position $X'_0 = 1$, $Y'_0 = S$. The method of our paper answers (in a sense) this question.

2. COUPLING

Rather than computing the probabilities for the OK Corral process directly, we couple it with Friedman's urn, and solve the relevant problem for the urn first.

We start the OK corral process by setting $X_0 = Y_0 = N$, let $\tau = \min\{t\colon X_t = 0 \text{ or } Y_t = 0\}$ and $S = X_\tau + Y_\tau$. From now on, we condition on the event $Y_\tau > 0$, so that $S \equiv Y_\tau$.

[3] In Kingman[7] the notations N denotes $X_t + Y_t$

[4] $a > 0$, $b = 0$ corresponds to Pólya urn

Suppose that $Y_1 < Y_0$. Then for each path connecting (N, N) with $(0, S)$ there exist $k \geqslant 1$ and two integer-valued sequences $x_0, x_1, \dots, x_k$ and $y_0, y_1, \dots, y_k$ such that $x_{i-1} < x_i$, $y_{i-1} < y_i$ for all i with $x_0 = 0$, $y_0 = S$, $x_k = y_k = N$ and the process (X_t, Y_t) comes to $(0, S)$ following the path

$$\begin{aligned} &(x_k, y), \quad y = y_k, y_k - 1, \dots, y_{k-1}, \\ &(x, y_{k-1}), \quad x = x_k, x_k - 1, \dots, x_{k-1}, \\ &\qquad \vdots \\ &(x, y_0), \quad x = x_1, x_1 - 1, \dots, x_0. \end{aligned}$$

The probability of such a path equals

$$\frac{1}{(2N)(2N-1)\cdots(S+1)} \left(\prod_{i=1}^{k} x_i^{y_i - y_{i-1}} \right) \left(\prod_{i=1}^{k} y_{i-1}^{x_i - x_{i-1}} \right). \tag{2}$$

Now consider Friedman's urn with the transition probabilities (1). The probability that (X', Y') comes to (N, N) from $(0, S)$ following the path described above, moving backwards, equals

$$\frac{1}{S(S+1)\cdots(2N-1)} \left(\prod_{i=1}^{k} x_i^{y_i - y_{i-1}} \right) \left(\prod_{i=1}^{k} y_{i-1}^{x_i - x_{i-1}} \right). \tag{3}$$

Comparing the probabilities (2) and (3) we observe that

$$\begin{aligned} \mathsf{P}(\swarrow) &:= \mathsf{P}((X, Y) \text{ hits } (0, S) \mid (X_0, Y_0) = (N, N)) \\ &= \frac{S}{2N} \mathsf{P}((X', Y') \text{ hits } (N, N) \mid (X'_0, Y'_0) = (0, S)) =: \frac{S}{2N} \times \mathsf{P}(\nearrow) \end{aligned} \tag{4}$$

actually for *any* path connecting $(0, S)$ and (N, N) (the case $X_1 < X_0$ can be analyzed in the same way). Note that the relation above also holds for paths connecting arbitrary two points, not necessarily $(0, S)$ and (N, N). Therefore, to calculate $\mathsf{P}(\swarrow)$ it suffices to compute $\mathsf{P}(\nearrow)$. This is what we do next.

3. DECOUPLING

As mentioned before, Freedman's results are not applicable for the speed of the convergence, while this is what we actually need (to compute asymptotically $\mathsf{P}(\nearrow)$ when N and S are large).

To this end, we will use a procedure known as *Rubin's construction* due to Herman Rubin introduced in Davis[3] for the study of reinforced random walks. Later his method was also applied to a variety of problems, ranging from rather theoretical ones, mostly related to reinforced random walks, (see Sellke[10] and Limic[8]) to the applied ones (see Khanin and Khanin[6]).

The essence of this construction is extremely simple and in our case runs as follows. Consider two *independent* birth processes, U_t and V_t, where t is continuous, both with the transition rate $\lambda_k\, dt = P(U_{t+dt} = U_t + 1 \mid U_t = k) = 1/k\, dt$ (the same formula holds for V_t). Set $U_0 = 1$ and $V_0 = S$. From the properties of independent exponential random variables it follows that the process (U_t, V_t) considered at the times when either of its coordinates changes, has the same distribution as the Friedman's urn (X', Y') described above. Now the independence of the coordinates of (U_t, V_t) allows us to compute desired distributions.

On a side, we would like to mention that Rubin's construction is apparently related to the Athreya and Karlin[1] embedding of the urn in a single global process.

4. COMPUTING $\mathsf{P}(\nearrow)$

For an integer n let $W_n = \inf\{t: U_t = n\}$ ($W'_n = \inf\{t: V_t = n\}$ resp.) be the nth waiting time for the process U (V resp.). Since $W_{i+1} - W_i$ are independently exponentially distributed, W_n and W'_n can be represented as

$$W_n = \sum_{k=1}^{n-1} k\xi_k, \qquad W'_n = \sum_{k=S}^{n-1} k\zeta_k, \tag{5}$$

where ξ_k's and ζ_k's ($k = 1, 2, \ldots$) are IID exponential random variables with rate 1. The process (U_t, V_t) passes through the point (N, N) if and only if $W_N \in [W'_N, W'_{N+1})$ or $W'_N \in [W_N, W_{N+1})$.

Let us compute the probability of the first event. To simplify notations, set $A = W_S$, $B = W_N - W_S$ and $C = W'_N$. We have

$$\mathsf{P}(W_N \in [W'_N, W'_{N+1})) = \mathsf{P}(A - N\zeta_N < C - B \leqslant A). \tag{6}$$

Note that $B - C$ and ζ_N are independent of A. Hence, integrating by parts,

$$\begin{aligned} \mathsf{P}(A - N\zeta_N < (C-B) \leqslant A \mid A) &= \int_0^\infty \frac{1}{N} e^{-z/N}\, dz \int_{A-z}^{A} f_{C-B}(x)\, dx \\ &= \int_0^\infty e^{-z/N} f_{C-B}(A-z)\, dz, \end{aligned} \tag{7}$$

where $f_{C-B}(x)$ is the pdf of the random variable $C - B$.

Next, we need

Lemma 1. As $S \to \infty$,

$$\frac{A - S(S-1)/2}{\sqrt{S^3/3}} \xrightarrow{\mathscr{D}} \mathcal{N}(0, 1). \tag{8}$$

Also, when both $N \to \infty$ and $S \to \infty$ such that $S = o(N)$

$$\frac{C - B}{\sqrt{2N^3/3}} \xrightarrow{\mathscr{D}} \mathcal{N}(0, 1). \tag{9}$$

Proof. Immediately follows from the representation (5), CLT with Lyapunov conditions (see Billingsley,[2] p. 312), and the well-known formula for $\sum_{i=1}^{n-1} i^p$, $p = 2, 3$, and 4. □

Now we want to strengthen (9). From now on we assume that $S = o(N)$.

Lemma 2. Let $f_N(x)$ be the pdf of $(C-B)/\sqrt{2N^3/3}$ and $f(x)$ be the pdf of $\mathcal{N}(0, 1)$. Then for any $\epsilon > 0$

$$\sup_x |f_N(x) - f(x)| = o(N^{-(1-\epsilon)}).$$

Proof. By the inversion formula,

$$f_N(x) - f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx}(\phi_N(t) - e^{-t^2/2})\, dt, \tag{10}$$

where

$$\phi_N(t) = \int_{-\infty}^{\infty} e^{itx} f_N(x)\, dx = \prod_{k=S}^{N-1} \frac{1}{1 + \dfrac{k^2 t^2}{2N^3/3}},$$

since $C - B = \sum_{k=S}^{N-1} k\zeta_k - k\xi_k$. Consequently, $\phi_N(t)$ can be written as

$$\phi_N(t) = \exp\left(-\frac{t^2}{2} + \frac{9}{40}\frac{t^4}{N} - \frac{9}{56}\frac{t^6}{N^2} + \cdots + o(t^2 + t^4/N + \cdots)\right) \tag{11}$$

or

$$\phi_N(t) = \frac{1}{1+\frac{t^2}{2}+\frac{t^4}{8}+\cdots+o(t^2+t^4/N+\cdots)}. \tag{12}$$

Let us split the area of integration in (10) into two parts: $[-N^\delta, N^\delta]$ where $0<\delta<\min(1/2, \epsilon/5)$, and its compliment. Then

$$\left|\int_{-N^\delta}^{N^\delta} e^{-itx}(\phi_N(t)-e^{-t^2/2})\,dt\right| \leqslant \frac{9}{40N}\int_{-N^\delta}^{N^\delta} |e^{-itx-t^2/2}|\, t^4(1+o(1))\,dt$$
$$= O(N^{-(1-5\delta)}) \tag{13}$$

using (11) and

$$\left|\int_{t\notin[-N^\delta, N^\delta]} e^{-itx}(\phi_N(t)-e^{-t^2/2})\,dt\right| \leqslant \int_{t\notin[-N^\delta, N^\delta]} \left\{\frac{1}{1+\frac{t^2}{2}+\cdots}+e^{-\frac{t^2}{2}}\right\} dt$$
$$= o(N^{-1}) \tag{14}$$

using (12). Combining (13) and (14), from the formula (10) we deduce that

$$|f_N(x)-f(x)| = O(N^{-(1-5\delta)})+o(N^{-1}) = o(N^{-(1-\epsilon)})$$

uniformly in x. □

From Lemma 2 it follows that for any $\epsilon>0$

$$f_{C-B}(y) = \frac{\exp\left(-\frac{y^2}{4N^3/3}\right)}{\sqrt{4\pi N^3/3}}+o(N^{-2.5+\epsilon}).$$

Plugging this into (7), we have

$$\mathsf{P}(W_N\in[W'_N, W'_{N+1})\mid A) = \int_0^\infty e^{-z/N}\left[\frac{\exp\left((-\frac{(A-z)^2}{4N^3/3}\right)}{\sqrt{4\pi N^3/3}}+o(N^{-2.5+\epsilon})\right]dz$$
$$= \int_0^\infty \frac{\exp\left(-u-\frac{(A-Nu)^2}{4N^3/3}\right)}{\sqrt{4\pi N/3}}\,du+o(N^{-1.5+\epsilon}).$$

Under the assumption that A is of order $N^{3/2}$, the integral in the RHS equals

$$\int_0^{\log N} \frac{\exp\left(-u-\frac{(A-Nu)^2}{4N^3/3}\right)}{\sqrt{4\pi N/3}}\,du+O(N^{-1.5})$$
$$=\frac{\exp\left(-\frac{3A^2}{4N^3}\right)}{\sqrt{4\pi N/3}}\left(1-\frac{1}{N}\right)\left(1+O\left(\frac{\log N}{N^{1/2}}\right)\right)+O(N^{-1.5})$$
$$=\frac{e^{-\frac{3A^2}{4N^3}}}{\sqrt{4\pi N/3}}+O\left(\frac{\log N}{N}\right),$$

whence

$$\mathsf{P}(W_N\in[W'_N,W'_{N+1})\mid A)=\frac{e^{-\frac{3A^2}{4N^3}}}{\sqrt{4\pi N/3}}+O\left(\frac{\log N}{N}\right). \tag{15}$$

Next we want to refine (8).

Lemma 3.

$$\sup_x \left|\mathsf{P}\left(\frac{A-S(S-1)/2}{\sqrt{S^3/3}}\leqslant x\right)-\Phi(x)\right|\leqslant\frac{\text{const}}{\sqrt{S}}, \tag{16}$$

where $\Phi(x)$ is the cdf of the standard normal distribution.

Proof. This result follows, e.g., from Theorem V.3.6 in Petrov.[9] □

Now suppose that S is of order $N^{3/4}$, and let

$$\rho:=\frac{\sqrt{3}\,S^2}{N^{3/2}}.$$

Lemma 3 yields

$$\mathsf{P}(|A-S^2/2|\geqslant S^{3/2}\log S)\leqslant\Phi(-\log S)+(1-\Phi(\log S))+O(S^{-1/2})$$
$$=O(S^{-1/2}),$$

hence with a large probability $A=O(S^2)=O(N^{3/2})$ which is consistent with our previous assumptions. Using (15) we conclude that $\mathsf{P}(W_N\in[W'_N,W'_{N+1}))$ equals

$$\mathsf{P}\left(W_N \in [W'_N, W'_{N+1})| \left|A - \frac{S^2}{2}\right| \leqslant S^{\frac{3}{2}} \log S\right) + O\left(\frac{1}{(SN)^{1/2}} + \frac{\log N}{N}\right)$$
$$= \frac{\exp\{-3[S^2/2 + O(S^{\frac{3}{2}} \log S)]^2/(4N^3)\}}{\sqrt{4\pi N/3}} + O(N^{-3/8-1/2})$$
$$= \frac{e^{-\rho^2/16}}{\sqrt{4\pi N/3}}\left[1 + O\left(\frac{\log N}{N^{3/8}}\right)\right]. \quad (17)$$

The formula for $\mathsf{P}(W'_N \in [W_N, W_{N+1}))$ can be obtained similarly, and the expression for it also equals the RHS of (17) because $\{W'_N \in [W_N, W_{N+1})\} = \{A \leqslant C - B < A + N\xi_N\}$ and $N\xi_N \overset{D}{=} N\zeta_N$ and $N\xi_N = o(A)$ a.s.

Hence,

$$\mathsf{P}(\nearrow) = \mathsf{P}(W_N \in [W'_N, W'_{N+1})) + \mathsf{P}(W'_N \in [W_N, W_{N+1}))$$
$$= \sqrt{\frac{3}{\pi N}}\, e^{-\rho^2/16}\left[1 + O\left(\frac{\log N}{N^{3/8}}\right)\right],$$

implying by (4)

$$\mathsf{P}(\swarrow) = \sqrt{\frac{\rho}{4\pi\sqrt{N/3}}}\, e^{-\rho^2/16} + O\left(\frac{\log N}{N^{9/8}}\right) = \frac{\rho}{2S\sqrt{\pi}} e^{-\rho^2/16} + O\left(\frac{\log S}{S^{3/2}}\right). \quad (18)$$

5. RESULTS AND DISCUSSION

Recall that throughout the most of previous arguments we supposed that $X_\tau = 0$, and now it is time to use the symmetry between X and Y. Therefore, if the number of the gunmen on each side at the beginning is N, then the probability that the number of the survivals after one of the groups is completely eliminated is S equals

$$= \frac{\rho}{S\sqrt{\pi}} e^{-\rho^2/16} + O\left(\frac{\log S}{S^{3/2}}\right)$$

by doubling the expression in (18).

Using the same technique it is not hard to see that if the number of the gunmen is not *exactly* equal at the beginning of the game, but at the same time the difference is not significant, namely $o(N^{1/2})$, the the formula above does not change with the exception of the correction term, which becomes

$$O\left(\frac{\log S}{S^{3/2}}\right) + O(|X_0 - Y_0|/N^{0.5}).$$

This can be obtained by noting that if $X_0 - Y_0 = o(N^{1/2})$ then the expression for A can be "corrected" by the term of order $o(N^{3/2})$. This is again consistent with Kingman.[7]

6. EXACT FORMULA

Note that the probability in (6) can be expressed as

$$\int_0^\infty e^{-z/N} f_{A+B-C}(z)\, dz, \tag{19}$$

where $f_{A+B-C}(z)$ is the density of random variable $A+B-C$, by arguments identical to the one applied in (7). To compute $f_{A+B-C}(z)$ we use Fourier transform:

$$\mathsf{E}\, e^{i\lambda(A+B-C)} = \prod_{k=1}^{N-1} \frac{1}{1-ik\lambda} \times \prod_{k=S}^{N-1} \frac{1}{1+ik\lambda} = \sum_{k=1}^{N-1} \frac{a_k}{1-ik\lambda} + \sum_{k=S}^{N-1} \frac{b_k}{1+ik\lambda}$$

for some a_k's and b_k's. Using standard techniques, we obtain that

$$a_k = \frac{(-1)^{N-k-1}\, k^{2N-S-2}(k+S-1)!}{(k-1)!\,(N-1+k)!\,(N-1-k)!}, \qquad k = 1, 2,..., N-1,$$

$$b_k = \frac{(-1)^{N-k-1}\, k^{2N-S-2} k!}{(k-S)!\,(N-1+k)!\,(N-1-k)!} \qquad k = S, S+1,..., N-1.$$

and therefore $f_{A+B-C}(z) = \sum_{k=1}^{N-1} a_k k^{-1} e^{-z/k} + \sum_{k=S}^{N-1} b_k k^{-1} e^{-z/k}$. Plugging this into (19) and computing the integral, we calculate the *exact* expression for $\mathsf{P}(W_N \in [W'_N, W'_{N+1}))$. Repeating this argument for $\mathsf{P}(W'_N \in [W_N, W_{N+1}))$ and using (4), we finally obtain that the probability that precisely S gunmen (on either side) will survive the shooting, provided there were initially N on both sides, is given by

$$\sum_{k=1}^{N-1} \frac{(-1)^{N-k-1}\, N k^{2N-S-2}(k+S-1)!}{(k-1)!\,(N+k)!\,(N-k-1)!} + \sum_{k=S}^{N-1} \frac{(-1)^{N-k-1}\, N k^{2N-S-2} k!}{(k-S)!\,(N+k)!\,(N-k-1)!}.$$

Unfortunately, since this is a sum with alternating signs, we are not able to get more insight on this probability than already provided in Section 5. Just for an illustration, we present the probability that exactly 5 gunmen will survive provided there were initially 20 on both sides:

$$\frac{1531045978341826356567228910022456158188 7}{2833039177944089355366705797208735744000 00} \approx 0.05404.$$

This number is computed using the analytical expression for the probability of survival of exactly S gunmen.

REFERENCES

1. Athreya, K., and Karlin, S. (1968). Embedding of urn schemes into continuous time Markov branching process and related limit theorems. *Ann. Math. Statist.* **39**, 1801–1817.
2. Billingsley, P. (1985). *Probability and Measure*, 2nd. edn., Wiley, New York.
3. Davis, B. (1990). Reinforced random walk. *Probab. Theory Related Fields* **84**, 203–229.
4. Freedman, D. (1965). Bernard Friedman's urn. *Ann. Math. Statist.* **36**, 956–970.
5. Friedman, B. (1949). A simple urn model. *Comm. Pure Appl. Math.* **2**, 59–70.
6. Khanin, K., and Khanin, R. (2001). A probabilistic model for establishing of neuron polarity. *J. Math. Biol.* **42**, 26–40.
7. Kingman, J. F. C. (1999). Martingales in the OK Corral. *Bull. London Math. Soc.* **31**, 601–606.
8. Limic, V. (2001). Attracting edge property for a class of reinforced random walks. *preprint*.
9. Petrov, V. V. (1975). *Sums of Independent Random Variables*, 2nd. edn., Springer, Berlin.
10. Sellke, T. (1994). *Reinforced Random Walk on the d-Dimensional Integer Lattice*, Department of Statistics, Purdue University, Technical report #94-26.
11. Williams, D., and McIlroy, P. (1998). The OK Corral and the power of the law (a curious Poisson kernel formula for a parabolic equation). *Bull. London Math. Soc.* **30**, 166–170.

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CORRECTION

Correction to: Solution to the OK Corral Model via Decoupling of Friedman's Urn

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In this note we acknowledge a mistake made in Section 6 of [1], for the exact probability for the number of survivors in the OK Corral model, given near the bottom of page 275.

Here is the correct (and, in fact, simpler) computation. Let ν be the (random) number of survivors on one of the sides, when the other side is exterminated. Observe that

$$\mathsf{P}(\nu \le S) = 2\mathsf{P}\left(\sum_{k=S+1}^{N} k\xi_k < \sum_{k=1}^{N} k\zeta_k\right) = 2\mathsf{P}(\eta_S > 0)$$

where the factor "2" comes from the fact that each side is equally likely to survive, and

$$\eta_S = \sum_{k=1}^{N} k\zeta_k - \sum_{k=S+1}^{N} k\xi_k.$$

For a λ sufficiently close to 0, we can compute the Laplace transform of η_S as

$$\varphi_{\eta_S}(\lambda) = \mathsf{E}\, e^{\lambda\eta_S} = \prod_{k=1}^{N} \frac{1}{1+\lambda k} \cdot \prod_{k=\sigma+1}^{N} \frac{1}{1-\lambda k} = \sum_{k=1}^{N} \frac{a_k}{1+\lambda k} + \sum_{k=\sigma+1}^{N} \frac{b_k}{1-\lambda k}$$

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where the constants

$$a_k = a_k(N, S) = \frac{(-1)^{N-k} k^{2N-S-1} (k+S)!}{(N-k)!(k-1)!(N+k)!}$$

are obtained using the partial fractions decomposition. Since we can invert the above Laplace transform and obtain that the density of η_S is given by

$$f_{\eta_S}(x) = \sum_{k=1}^{N} a_k \frac{e^{-x/k}}{k} \mathbf{1}_{\{x>0\}} + \sum_{k=\sigma+1}^{N} b_k \frac{e^{x/k}}{k} \mathbf{1}_{\{x<0\}},$$

we can compute

$$\mathsf{P}(\nu \le S) = 2\mathsf{P}(\eta_S > 0) = 2\int_0^\infty f_{\eta_S}(x)dx = 2\sum_{k=1}^{N} a_k(N, S).$$

Consequently, since $\mathsf{P}(\nu = S) = \mathsf{P}(\nu \le S) - \mathsf{P}(\nu \le S-1)$, we conclude that

$$\mathsf{P}(\nu = S) = 2\sum_{k=1}^{N} [a_k(N, S) - a_k(N, S-1)] = 2\,S \sum_{k=1}^{N} \frac{(-1)^{N-k} k^{2N-S} (k+S-1)!}{(N-k)!\,k!\,(N+k)!}$$

for $S = 1, 2, \ldots, N$.

Note that the formula at the bottom of page 275 for $\mathsf{P}(\nu = 5)$ is actually correct.

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Reference

1. Kingman, J.F.C., Volkov, S.E.: Solution to the OK Corral model via decoupling of Friedman's urn. J. Theor. Probab. **16**(1), 267–276 (2003)