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# $p$-adic probability interpretation of Bell's inequality ☆

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**Abstract**

We study the violation of Bell's inequality using a $p$-adic generalization of the theory of probability. $p$-adic probability is introduced as a limit of relative frequencies but this limit exists with respect to a $p$-adic metric. In particular, negative probability distributions are well defined on the basis of the frequency definition. This new type of stochastics can be used to describe hidden-variables distributions of some quantum models. If the hidden variables have a $p$-adic probability distribution, Bell's inequality is not valid and it is not necessary to discuss the experimental violations of this inequality.

## 1. Introduction

The theory of hidden variables arisen from the Einstein–Podolsky–Rosen (EPR) [1] "realistic philosophy" and other theories based on hidden-variables ideas [2–5] are very attractive. However, one of the main problems of the EPR hidden-variable description is the contradiction with quantum mechanics predictions regarding Bell's inequality [6] and its generalizations [7]. Experimental tests on the basis of Bell-type inequalities show that there exist quantum models, where the hidden-variables description has no meaning.

We wish to apply to Bell's inequality a mathematical apparatus developed inside of new physical (at the moment, only theoretical) models, $p$-adic quantum physics [8–10] [2]. $p$-adic phys models are attempts to describe reality with the aid of a number field $\mathbb{Q}_p$ which has many properties different from the real or complex case. It is relevant to note that we will not consider any model of $p$-adic physics. We will use only the corresponding mathematical methods.

Quantum mechanics with wave functions assuming values in $\mathbb{Q}_p$ is one of the numerous $p$-adic models. It was developed in Ref. [12]. The main problem of this theory is the probability interpretation of these wave functions. A new mathematical theory, the $p$-adic valued theory of probability, was proposed in Refs. [13,9] to resolve this problem. As usual we consider a probability as the limit of relative frequencies $\nu_n$ but with respect to another metric on the field of rational numbers $\mathbb{Q}$. The following natural idea is the basis of our constructions. The physical numbers are rational numbers. We can ob-

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[2] For $p$-adic number fields $\mathbb{Q}_p$, see Ref. [11].


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tain in any experiment only finite fractions and there are no real or complex experimental data. Then we can study these rational data with the aid of different mathematical methods. $p$-adics helps us to find some additional information about these rational numbers which we cannot find using real numbers. In particular, there exist random sequences in which the $\nu_n$ oscillate between 0 and 1 with respect to the usual real metrics but are stable with respect to one of the $p$-adic metrics [13,9]; in Refs. [13,9] also the results of computer simulation are given. From the usual point of view such sequences are chaotic, and it would be impossible to define real probabilities. But there is a well defined $p$-adic probability distribution.

$p$-adic probability has many of the properties of the standard real probability: additivity, conditional probabilities, independent events, etc. [9]. But one property is very different from the ordinary case. All probabilities of the usual theory belong to the segment [0,1] of $\mathbb{R}$, but it was shown in Refs. [13,9] that we can represent every rational number as the limit of $\nu_n$ with respect to a $p$-adic metric. In particular, every negative rational number can be represented as the limit of relative frequencies. This fact is the foundation of our further considerations. Negative probability distributions are an ordinary object in the $p$-adic theory of probability. We have a statistical connection between the relative frequencies and negative probabilities. This is impossible in the usual probability theory.

## 2. $p$-adic numbers

The field of real numbers $\mathbb{R}$ is constructed as a completion of the field of rational numbers $\mathbb{Q}$ with respect to the metric $\rho(x, y) = |x - y|$, where $|\ |$ is the usual absolute-value norm. Fields of $p$-adic numbers $\mathbb{Q}_p$ are constructed in the same way. There is an infinite sequence of $p$-adic number fields, every prime number $p = 2, 3, 5, \ldots$ has its own field. A $p$-adic norm $|\ |_p$ is defined in the following way. First, we define it for natural numbers.

Every natural number $n$ can be represented as the product of powers of prime numbers: $n = 2^{r_2} 3^{r_3} \ldots p^{r_p} \ldots$ . Then $|n|_p = p^{-r_p}$, by definition $|0|_p = 0$, $|-n|_p = |n|_p$, and $|n/m|_p = |n|_p / |m|_p$. The completion of $\mathbb{Q}$ with respect to the metric $\rho_p(x, y) = |x - y|_p$ is a locally compact field $\mathbb{Q}_p$.

It is an intrinsic fact of the theory of numbers [11] that the only possibility to introduce a norm on $\mathbb{Q}$ is to use a real one $|\ |$ or a $p$-adic one.

Any $p$-adic number can be represented in a unique manner in the form of a (convergent) series in powers of $p$,

$$x = a_{-n}/p^n + \ldots a_0 + \ldots + a_k p^k + \ldots, \qquad (1)$$

where $a_j = 0, 1, \ldots, p - 1$, are digits of the $p$-adic expansion. In a real case such an expansion is infinite in the direction of negative degrees, in the $p$-adic case it is so in the direction of positive degrees. Expansion (1) is the basis of the $p$-adic statistical simulation [9,13]. In some sense the negative probabilities can arise in the same way as $-1 = 1 + 2 + 4 + \ldots$ and this series converges in $\mathbb{Q}_2$.

## 3. The foundations of the theory of probability and $p$-adic theory of probability

The frequency definition of probability proposed by von Mises in 1919 has played an important role in the construction of the foundations of modern probability theory. This definition exerted a strong influence on the measure-theoretical axioms of Kolmogorov [14]. As motivation of his axioms, Kolmogorov used the properties of the limits of relative frequencies [14]. We are interested in the manner in which Kolmogorov's axiom 2 arose; in accordance with this axiom, the probability of any event E is a nonnegative real number. In Ref. [14], Kolmogorov considers von Mises' definition [15] of probability as the limit of the relative frequencies of the occurrence of the event E. Further, since the relative frequencies are rational numbers that lie between zero and unity, their limits with respect to the real metric are real numbers between zero and unity.

It is probably necessary to recall in its general features the main propositions of von Mises' theory of probability [15].

A basic object of Mises' theory is a *collective*. Let $\mathcal{S}$ be a random experiment and $\mathrm{T} = \{\alpha_1, \ldots, \alpha_m\}$ be the set of all possible realizations of

$\mathcal{S}$ (in the simplest case $\mathrm{T}=\{0, 1\}$). An infinite sequence of realizations of $\mathcal{S}$,

$$x=(x_1,\ldots, x_n,\ldots), \quad x_i \in \mathrm{T}, \tag{2}$$

is said to be a collective if for every label $\alpha \in \mathrm{T}$ there exists a limit of relative frequencies $\nu_n(\alpha)=k(\alpha)/n$: $\mathrm{P}(\alpha)=\lim_{n\to\infty} \nu_n(\alpha)$. This limit is said to be a probability of $\alpha$.

Now a sequence (2) is said to be a $p$-adic collective [9,13] if the limit of relative frequencies exists with respect to the $p$-adic metric for every label $\alpha \in \mathrm{A}$. This limit $\mathrm{P}_p(\alpha)=\lim_{n\to\infty} \nu_n(\alpha)$ is called a $p$-adic probability. In Refs. [9,13], $p$-adic collectives were considered from the theoretical and computer points of view. A series of computer statistical experiments was proposed. The relative frequencies oscillated between 0 and 1 with respect to the real metric but stabilize very quickly with respect to one of the $p$-adic metrics ($p$ is a parameter of the model).

A measure theoretical definition of the $p$-adic probability is generated by the frequency definition. It is a $\mathbb{Q}_p$-valued normalized measure (see the paper by Schikhof [11]).

## 4. $p$-adic stochastic approach to Bell's inequality

First, let us consider a standard hidden-variable approach to Bell's inequality. Let $\Lambda$ be a set of hidden variables $\lambda$ distributed with a (usual) probability density $\rho(\lambda)$. Then from the mathematical point of view all problems are contained in the integral

$$\langle AB\rangle = \int_\Lambda A(\lambda)B(\lambda)\, \mathrm{d}\mu(\lambda), \tag{3}$$

where $A(\lambda)$ and $B(\lambda)$ are random variables with values $+1$ or $-1$ and $\mathrm{d}\mu(\lambda)=\rho(\lambda)\,\mathrm{d}\lambda$. The estimate of such mean values is based on two properties of probability: $\rho(\lambda)$ is nonnegative and $\mu(\Lambda)=1$.

Now let us suppose that hidden variables $\lambda$ are distributed with a $p$-adic probability law. Thus $\mu$ is a $\mathbb{Q}_p$-valued measure on $\Lambda$, $\mu(\Lambda)=1$. $A(\lambda)$ and $B(\lambda)$ are random variables with respect to $\mu$ assuming the same values $+1$ or $-1$. Then we cannot as usual estimate (3) and Bell's inequality is no more valid for a hidden-variables distribution.

This is why, if we suppose that a physical model has a $p$-adic hidden-variable distribution (in this case the relative frequencies for hidden variables stabilize in $p$-adic metrics), there is no contradiction anymore between quantum mechanical experiments and hidden-variables description.

## 5. Hidden-variables stochastics

Our main idea is that there are two types of stochastics. The first one is the quantum mechanics stochastics. It is described by the usual theory of probability in all known physical models. This stochastics probably reflects a macrostructure of our equipment. Then we suppose that the internal structure of some physical models can be described by means of another stochastics, the $p$-adic one. Of course, there exist quantum models with the ordinary stochastic structure. Bell's inequality is valid for such models.

Thus the only implication of violations of Bell's inequality in some quantum models is that the hidden-variables of these models are $p$-adic distributed.

Is it possible to get a $p$-adic type Bell's inequality? Can we get some kind of Bell's estimate of (3) in the $p$-adic distributed case? Using properties of $\mathbb{Q}_p$-valued measures [11], we obtain

$$\left|\int_\Lambda A(\lambda)B(\lambda)\right|_p \leqslant \|\mu\|_p, \tag{4}$$

where $\|\mu\|_p$ is a $p$-adic variation of the measure $\mu$. But this variation may be arbitrarily large also in the case of the normalized measure $\mu(\Lambda)=1$. This is why we need information on the variation of the $p$-adic probability distribution $\mu$.

What about the topological structure of a set $\Lambda$ of hidden variables? The theory of $p$-adic valued measures is well defined in the case when $\Lambda$ is a zero-dimensional topological space. For example, it can be one of the $p$-adic numbers fields $\mathbb{Q}_{p1}$, where $p_1$ can be different from $p$ (the hidden-variables distribution takes values in $\mathbb{Q}_p$). We can suppose that in the case of Bell's inequality violations a set of hidden variables $\Lambda$ has a zero-dimensional structure. Thus we have an example of $p$-adic physics [8,9]. Probably it would be possible to apply $p$-adic models to new types of hidden-variables theory.

## 6. Mixing of $p$-adic and real stochastics in experiments

Following Bohm we understand that "the so-called observables are not properties belonging to the observed system alone, but instead potentialities whose precise development depends just as much on the observing apparatus as on the observed system". The $p$-adic statistics of the microworld hidden-variables distribution is perturbed by the usual statistics of the apparatus. This is why we do not obtain $p$-adic random variables $A(\lambda)$, $B(\lambda)$ distributed with the aid of a $p$-adic probability $d\mu(\lambda)$ in physical experiments but mixed real-$p$-adic random variables $A(\lambda, \omega), B(\lambda, \omega)$. If as usual we suppose that the hidden variables $(\lambda)$ distribution $d\mu(\lambda)$ is independent of the distribution of external stochastics, we get a product $dM(\lambda,\omega) = d\mu(\lambda)\, d\nu(\omega)$ where $d\nu(\omega)$ is a usual probability measure. It would be a difficult problem to extract a $p$-adic stochastics from this mixed distribution.

## 7. Example of a $p$-adic probability distribution with rational values and negative probabilities of some events

Let $\Omega$ be the standard Bernoulli probability space, the space of sequences $\omega = (\omega_j)$, $\omega_j = 0, 1$. Let $\mathrm{I} = \cup \mathrm{I}_n$, where $\mathrm{I}_n$ is the set of all vectors of length $n$ with coordinates 0, 1. Let $i \in \mathrm{I}_n$ and $\mathrm{B}_i = \{\omega \in \Omega: \omega_1 = i_1, \ldots, \omega_n = i_n\}$ is a cylindrical subset. Then the standard Bernoulli probability is defined by $\mu(\mathrm{B}_i) = 1/2^n$. It is an additive set-function and it can be extended to the standard Bernoulli probability $\mu_\infty$ on the $\sigma$-algebra generated by $\{\mathrm{B}_i\}$.

We are interested in another extension of $\mu$. As $\mu$ assumes its values in $\mathbb{Q}$, it is also possible to consider it as a $\mathbb{Q}_p$-valued measure. There are some restrictions on the value of $p$, $p \neq 2$. For example, it is possible to choose $p = 3$. $\mu$ can be extended to a bounded $\mathbb{Q}_3$-valued measure $\mu_3$, 3-adic probability. We note that $\Omega$ is isomorphic to the unit ball $\mathrm{U}_1(0) = \{x \in \mathbb{Q}_2: |x|_2 \leqslant 1\}$ of $\mathbb{Q}_2$, see 2-adic expansion (1), $x \in \mathrm{U}_1(0)$ iff there is no negative degrees of 2 in (1) and $x \to \omega$, $\omega_1 = x_0, \ldots, \omega_n = x_{n+1}, \ldots$. According to this isomorphism $\mathrm{B}_i$ is a ball $\mathrm{U}_{2^{-n}}(a) = \{x \in \mathbb{Q}_2: |x - a|_2 \leqslant 2^{-n}\}$, where $a$ is an arbitrary point of $\mathbb{Q}_2$ with the property $a_0 = i_1, \ldots, a_{n-1} = i_n$.

It is interesting that the probabilities $\mu_\infty$ and $\mu_3$ coincide for all events which depend of a finite number of experiments $(\mathrm{B}_i)$. Thus, we cannot distinguish these two distributions in experiment.

Now let us introduce on $\Omega$ a density $\rho: \Omega \to \mathbb{Q}$ to generate a new probability distribution. Set $i_{n0} = (0, \ldots, 0) \in \mathrm{I}_n$ and $\mathrm{S}_n = \mathrm{B}_{i_{n0}} \setminus \mathrm{B}_{i_{(n+1)0}}$. Then

$$\Omega = \bigcup_{n=0}^{\infty} S_n, \quad S_n \cap S_m = \emptyset, \quad n \neq m. \tag{5}$$

Set $\rho(\lambda) = 3^n$ on $\mathrm{S}_n$. Let us try to normalize a measure $d\nu(\lambda) = \rho(\lambda)\, d\mu(\lambda)$. We have in a real case

$$\int_\Omega d\nu_\infty(\lambda) = \int_\Omega \rho(\lambda)\, d\mu_\infty(\lambda) = \infty.$$

Thus, it is impossible to normalize this distribution and there is no standard probability distribution.

Now let us consider the 3-adic case. We note that $\mathrm{S}_n = \{x \in \mathbb{Q}_2: |x|_2 = 2^{-n}\}$ is a 2-adic sphere of radius $2^{-n}$ with the center in zero. Representation (5) is nothing but the representation of the unit 2-adic ball as a union of spheres. We get

$$\delta = \int_\Omega d\nu_3(\lambda) = \sum_{n=0}^{\infty} 3^n(1/2^n - 1/2^{n+1}).$$

This series converges in $\mathbb{Q}_3$, as $|(3/2)^n|_3 = 1/3^n \to 0$ and in the $p$-adic case the series $\Sigma\, a_n$ converges iff $|a_n|_p \to 0$. We get $\delta = -1$. Thus we can normalize the density $\rho(\lambda) \to \rho_{\mathrm{norm}}(\lambda) = \rho(\lambda)/\delta$ and introduce the 3-adic valued distribution $dP_3(\lambda) = \rho_{\mathrm{norm}}(\lambda)\, d\mu_3(\lambda)$. The 3-adic probabilities of events $\mathrm{S}_n$ are negative: $P_3(\mathrm{S}_n) = -3^n/2^{n+1}$ but the probabilities of some events are positive: for $\mathrm{B}_{i_{n0}} = \bigcup_{k=n}^{\infty} \mathrm{S}_k$ we get $P_3(B_{i_{n0}}) = \Sigma_{k=n}^{\infty} P_3(\mathrm{S}_k) = (3/2)^n$.

How would it be possible to impress the probability distribution $dP_3(\lambda)$ in the intuitive way? Let us realize $\Omega$ as the segment [0,1], $i: \Omega \to \mathbb{R}$, $\omega \to s = \Sigma_{j=1}^{\infty} \omega_j 2^{-j}$. Then $i(\mathrm{S}_0) = [1/2, 1]$, $i(\mathrm{S}_1) = [1/4, 1/2], \ldots, i(\mathrm{S}_n) = [1/2^{n+1}, 1/2^n], \ldots$ ($i$ is not an isomorphism, for example, $i((1,0,\ldots,0,\ldots)) = i((0,1,\ldots,1,\ldots)) = 1/2$). The probability density $\rho(\lambda)$ on [0,1] has the form $\rho(\lambda) \equiv 3^n$ on the segment $[1/2^{n+1}, 1/2^n]$. Thus, this probability distribution will be very quickly concentrated in neighbourhoods of zero, so quickly that real numbers cannot describe this.

The EPR paradox is not the only physical formalism where negative probability distributions can play a great role. Negative probability distributions were first introduced by Dirac [16] in the formalism of the relativistic quantization of photons (it was the price for the negative energies). Probably $p$-adic stochastics is also a realistic realization of Dirac's hypothetical world.

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## References

[1] A. Einstein, B. Podolsky and N. Rosen, Phys. Rev. 47 (1935) 777.
[2] L. de Broglie, La physique quantique: restera-t-elle indeterministe (Gauthiers Villars, Paris, 1953).
[3] D. Bohm and J.P. Vigier, Phys. Rev. 96 (1954) 208.
[4] H. Freistadt, Nuovo Cimento Suppl. 5 (1957) 1.
[5] J.L. Andrade e Silva and G. Lochak, Quanta (McGraw-Hill, New York, 1969).
[6] J.S. Bell, Physics 1 (1965) 195.
[7] J.F. Clauser and A. Shimony, Rep. Prog. Phys. 41 (1978); D. Home and F. Selleri, Nuovo Cimento Riv. 14 (1991).
[8] V.S. Vladimirov, I.V. Volovich and E.I. Zelenov, $p$-adic numbers in mathematical physics (World Scientific, Singapore, 1993).
[9] A.Yu. Khrennikov, $p$-adic valued distributions in mathematical physics (Kluwer, Dordrecht, 1994).
[10] I.V. Volovich, Classical Quantum Grav. 4 (1987) L83; I.V. Volovich, Number theory as the ultimate physical theory, preprint TH.4781/87 (1987); P.G.O. Freund and M. Olson, Phys. Lett. B 199 (1987) 186; P.G.O. Freund, M. Olson and E. Witten, Phys. Lett. B 199 (1987) 191; P.H. Frampton and Y. Okada, Phys. Rev. Lett. 60 (1988) 484.
[11] K. Mahler, Introduction to $p$-adic numbers and their functions, Cambridge tracts in mathematics, Vol. 64 (Cambridge Univ. Press, Cambridge, 1973); W. Schikhof, Ultrametric calculus, Cambridge studies in advanced Mathematics, Vol. 4 (Cambridge Univ. Press, Cambridge, 1984); Z.I. Borevich and I.R. Schafarevich, Number theory (Academic Press, New York, 1966).
[12] A.Yu. Khrennikov, Russ. Math. Surv. 45 (1990) 87; J. Math. Phys. 32 (1991) 932; Theor. Math. Phys. 83(1990) 124; Sov. Phys. Dokl. 35 (1990) 867.
[13] A.Yu. Khrennikov, Sov. Phys. Dokl. 37 (1992); Acad. Sci. Dokl. Math. 47 (1993) 477; Statistical interpretation of $p$-adic quantum mechanics, preprint No. 16, Institute of General Physics, Russian Academy of Sciences, Moscow (1992) [in Russian]; Probability interpretation of the $p$-adic valued quantum theories, in: Abstracts, Symp. on The foundations of modern physics, Helsinki, June 1994, p. 134; R. Cianci and A.Yu. Khrennikov, Phys. Lett. B 328 (1994) 109.
[14] A.N. Kolmogoroff, Grundbegriffe der Wahrscheinlichkeitsrechnung (Berlin, 1933) [English translation (New York, 1950).
[15] R. von Mises, Math. Z. 5 (1919) 52; The mathematical theory of probability and statistics (Academic Press, New York, 1964).
[16] P.A.M. Dirac, Proc. R. Soc. A 180 (1942) 1.