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# Notes, Comments, and Letters to the Editor

## A Note on Stochastic Dominance and Inequality Measures*

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A sequence of partial orders (called inverse stochastic dominances) is introduced on the set of distribution functions (of nonnegative random variables). The partial orders previously defined are used to rank income distributions when Lorenz ordering does not hold, i.e., when Lorenz curves intersect. It is known that the Gini index is coherent with second degree stochastic dominance (and with second degree inverse stochastic dominance). It will be shown that it is coherent with third degree inverse stochastic dominance, too. It will finally be shown that a sequence of ethically flexible Gini indices due to D. Donaldson and J. A. Weymark (Ethically flexible Gini indices for income distribution in the continuum, *J. Econ. Theory* **29** (1983), 353–356) is coherent with the sequence of $n$th degree inverse stochastic dominances, *Journal of Economic Literature* Classification Numbers: 024, 225.


### 1. INTRODUCTION

The Lorenz curve has been widely used as a tool for ordering income distributions. The distributions $F, G$ are ordered by the Lorenz ordering ($F \geqslant_L G$) if the Lorenz curve of $F$ is nowhere below the Lorenz curve of $G$. The Lorenz ordering is only partial: two distributions are not comparable (under $\geqslant_L$) whenever their Lorenz curves intersect. For a review of this topic see Yitzhaki and Olkin [19]. Under the assumption of equality of the

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means, the Lorenz ordering is equivalent to second degree stochastic dominance (Atkinson [1]).

An order $\geqslant_A$ is finer than another partial order $\geqslant_B$ if $F \geqslant_B G$ implies $F \geqslant_A G$, i.e., $\geqslant_A$ orders all the distributions that $\geqslant_B$ orders. An order is linear if it orders every pair of distributions. An inequality measure $I$ is a functional of the distribution that induces a linear order $\geqslant_I$ in this way:

$$F \geqslant_I G \quad \text{iff} \quad I(F) \leqslant I(G).$$

If $I$ induces an order $\geqslant_I$, and $\geqslant_I$ is finer than $\geqslant_A$, then $I$ is said to be coherent with $\geqslant_A$.

In order to compare (in terms of inequality) pairs of distributions that are not ordered by Lorenz ordering, it is wise to choose (partial or linear) orders that are finer than Lorenz ordering, and therefore to choose inequality measures that are coherent with $\geqslant_L$. The sequence of $n$th degree stochastic dominances ($n$-SD) is a sequence of progressively finer partial orders (see Fishburn [7, 8] and Rolski [14]). Therefore, when the means of the distributions are equal, they are finer than the Lorenz ordering (for $n \geqslant 2$).

In this paper we will define a sequence of progressively finer partial orders of distributions, called $n$th degree inverse stochastic dominances ($n$-ISD). For $n = 1, 2$, $n$-ISD is equivalent to the corresponding $n$-SD. This equivalence does not hold for $n > 2$ (Shorrocks and Foster [15]). We prove a necessary condition for $n$-ISD, which involves the moment of the $(n-1)$st (out of $n-1$) order statistics. This result will then be used in proving some coherence results. First of all, it will be shown that Gini's index is coherent with third degree inverse stochastic dominance, and not only with second degree inverse stochastic dominance (a case that is well known). Then a class of ethically flexible indices, introduced by Donaldson and Weymark [4, 5] and called $S$-Ginis, will be examined, and it will be proved that the index of order $k$ is coherent with $n$-ISD for each $n \leqslant k+1$.

We want to emphasize that all the orders that we have defined are partial, while the orders induced by inequality measures are obviously linear. We consider this fact an advantage of the stochastic-dominance approach over the inequality-measure one, in that it allows one to distinguish between pairs of distributions that are actually orderable according to some criterion and pairs of distributions that are orderable only by comparing some particular functional of theirs. On the other hand, if a synthetic measure of inequality is actually required, it is interesting to know that it is at least coherent with some established partial order. The choice of the measure may be dictated by the choice of a partial order, with which it has to be coherent, and of some other axiomatic properties.

All the results are expressed in terms of distribution functions, and no hypothesis of absolute continuity or discreteness is required.

## 2. A Sequence of Stochastic Orders

Let $\mathscr{F}$ be the class of distribution functions (d.f.'s) on $[0, +\infty)$. Let $F \in \mathscr{F}$; we define

$$F^{-1}(y) = \inf\{x: F(x) \geqslant y\} \qquad 0 \leqslant y \leqslant 1.$$

This is the left-continuous version of the inverse of $F$. We define, for $F \in \mathscr{F}$,

$$F_n(x) = \int_0^x F_{n-1}(s)\, ds$$

$$F_1(x) = F(x).$$

It is well known that

$$F_n(x) = \frac{1}{(n-1)!} \int_0^x (x-y)^{n-1}\, dF(y) \qquad \forall x \geqslant 0.$$

Analogously, we define

$$F_n^{-1}(x) = \int_0^x F_{n-1}^{-1}(s)\, ds \qquad 0 \leqslant x \leqslant 1$$

$$F_1^{-1}(x) = F^{-1}(x).$$

The following lemma is easily proved by induction.

Lemma 1. *We have*

$$F_n^{-1}(x) = \frac{1}{(n-1)!} \int_0^x (x-y)^{n-1}\, dF^{-1}(y) \qquad 0 \leqslant x \leqslant 1. \tag{1}$$

*Proof.* The assertion is true for $n=1$. Assume that it is true for a generic $n$. Then

$$F_{n+1}^{-1}(x) = \int_0^x \frac{1}{(n-1)!} \int_0^t (t-y)^{n-1}\, dF^{-1}(y)\, dt$$

$$= \frac{1}{(n-1)!} \int_0^x \frac{(x-y)^n}{n}\, dF^{-1}(y),$$

where the last equality follows from Fubini's Theorem. ∎

Consider the two sequences of partial orders on $\mathscr{F}$. Let $F, G \in \mathscr{F}$:

$$F \geqslant_n G \quad \text{iff} \quad F_n(x) \leqslant G_n(x) \qquad \forall x \in \Re_+$$

$$F \geqslant_n^{-1} G \quad \text{iff} \quad F_n^{-1}(x) \geqslant G_n^{-1}(x) \qquad \forall x \in [0, 1]$$

The orders $\geqslant_n$ are the well known $n$th degree stochastic dominances (see Fishburn [7, 8, 9], Rolski [14]). We call the order $\geqslant_n^{-1}$ $n$th degree inverse stochastic dominance.

PROPOSITION 1. *Let $F, G \in \mathscr{F}$. If $F \geqslant_n^{-1} G$, then $F \geqslant_m^{-1} G$ for all $m > n$.*

*Proof.* If $F_n^{-1}(x) \geqslant G_n^{-1}(x)$ $\forall x \in [0, 1]$, then

$$\int_0^x F_n^{-1}(s)\, ds \geqslant \int_0^x G_n^{-1}(s)\, ds \qquad \forall x \in [0, 1],$$

that is, $F \geqslant_{n+1}^{-1} G$. Proceeding by induction, we obtain the result. ∎

It can be shown that $n$-SD and $n$-ISD are equivalent for $n = 1, 2$. When $n = 1$, the result is trivial; when $n = 2$, it can be proved by a slight generalization of an argument due to Atkinson [1], who proved the equivalence in the case of absolutely continuous distribution functions having the same mean. When $n \geqslant 3$ the equivalence does not hold anymore, as counterexamples in [3], [10], and [15] show.

The next theorem will provide necessary conditions for $n$-ISD. Let $(X_1 \wedge X_2 \wedge \cdots \wedge X_n) = \min_{i=1,\ldots,n} X_i$.

THEOREM 1. *If $F \geqslant_n^{-1} G$, then*

$$E(X_1 \wedge X_2 \wedge \cdots \wedge X_k) \geqslant E(Y_1 \wedge Y_2 \wedge \cdots \wedge Y_k)$$

*for all $k \geqslant n-1$, where $X_1, X_2, \ldots, X_k$ are i.i.d.r.v.'s distributed according to $F$, and $Y_1, Y_2, \ldots, Y_k$ are i.i.d.r.v.'s distributed according to $G$.*

*Proof.* If we compute (1) when $x = 1$, we have

$$\begin{aligned} F_n^{-1}(1) &= \frac{1}{(n-1)!} \int_0^1 (1-p)^{n-1}\, dF^{-1}(p) \\ &= \frac{1}{(n-1)!} \int_0^\infty (1-F(x))^{n-1}\, dx \\ &= \frac{1}{(n-1)!} E(X_1 \wedge X_2 \wedge \cdots \wedge X_{n-1}). \end{aligned}$$

See, e.g., David [2, p. 38]. If $F \geqslant_n^{-1} G$, namely $F_n^{-1}(x) \geqslant G_n^{-1}(x)$, $\forall 0 \leqslant x \leqslant 1$, then

$$E(X_1 \wedge X_2 \wedge \cdots \wedge X_{n-1}) \geqslant E(Y_1 \wedge Y_2 \wedge \cdots \wedge Y_{n-1}).$$

By Proposition 1,

$$F \geqslant_n^{-1} G \Rightarrow F \geqslant_k^{-1} G \qquad \forall k \geqslant n$$

and the result follows. ∎

## 3. Inequality Measures

We consider the definition of the Lorenz curve (Gastwirth [11])

$$L_X(p) = \frac{1}{E(X)} \int_0^p F^{-1}(z)\, dz \qquad 0 \leqslant p \leqslant 1,$$

where $E(X)$ is assumed to be finite. Let $X$, $Y$ be nonnegative r.v.'s with finite expectation. The Lorenz ordering $\geqslant_L$ is defined in the following way:

$$X \geqslant_L Y \qquad \text{iff} \quad L_X(p) \geqslant L_Y(p) \qquad \forall 0 \leqslant p \leqslant 1$$

(see Marshall and Olkin [12]).

Obviously, if $E(X) = E(Y)$, then

$$F_X \geqslant_2 F_Y \Leftrightarrow F_X \geqslant_2^{-1} F_Y \Leftrightarrow X \geqslant_L Y.$$

One of the most common measures of inequality is the Gini index $R$, which is defined as

$$R(F) = 1 - 2 \int_0^1 L_X(p)\, dp = 1 - \frac{2}{E(X)} \int_0^1 \int_0^p F^{-1}(t)\, dt\, dp$$
$$= 1 - \frac{2}{E(X)} F_3^{-1}(1) = 1 - \frac{E(X_1 \wedge X_2)}{E(X_1)}.$$

Therefore, whenever $F_X \geqslant_3^{-1} F_Y$ and $E(X) = E(Y)$, then

$$R(F_X) \leqslant R(F_Y). \tag{2}$$

This means that the Gini index is coherent with third degree inverse stochastic dominance when the r.v.'s have the same expectation. It was known that the Gini index is coherent with second degree stochastic dominance, hence with second degree inverse stochastic dominance (see Yitzhaki [17]). Our result is stronger, in that it insures that Inequality (2) holds for the Gini index, even when the Lorenz curves intersect, provided 3-ISD holds.

Donaldson and Weymark [4, 5] proposed two classes of single parameter indices (called $S$-Ginis) that generalize Gini's index (see also Yitzhaki [18]). For each $k \in \mathcal{N}$, they defined an absolute index

$$\Xi_k(F_X) = -\int_0^\infty x \, d(1 - F_X(x))^k$$

and a relative index

$$I_k(F_X) = 1 + \frac{\int_0^\infty x \, d(1 - F_X(x))^k}{E(X)}.$$

For the sake of simplicity, we consider only the case of $k$ a positive integer, whereas Donaldson and Weymark defined the indices for any real $k \geqslant 1$. The index $\Xi_k(F_X)$ (and therefore $I_k(F_X)$) exists whenever $E(X) < \infty$. Integrating by parts, we obtain

$$\Xi_k(F_X) = \int_0^\infty (1 - F_X(x))^k \, dx = E(X_1 \wedge \cdots \wedge X_k).$$

Therefore, by Theorem 1, if $F_X \geqslant_{n+1}^{-1} F_Y$, then $\Xi_k(F_X) \geqslant \Xi_k(F_Y)$ for $k \geqslant n$. If, furthermore, $E(X) = E(Y)$, then $I_k(F_X) \leqslant I_k(F_Y)$ for $k \geqslant n$.

It is evident that $I_2(F_X) = R(F_X)$ (see Dorfman [6], Yitzhaki [18]).

Resorting to the stochastic dominance results of Rolski [14] and Fishburn [7, 8], we can find a class of inequality measures coherent with $\geqslant_n^{-1}$ for any positive integer $n$.

THEOREM 2. *Let $\mathcal{M}_n$ be the class of functions $\phi$: $[0, 1] \to \mathfrak{R}$ such that*

$$\phi(x) = -\int_x^1 (s - x)^{n-1} \, d\tau(s)$$

*where $\tau$ is a positive measure. Then $F \geqslant_n^{-1} G$ if and only if*

$$\int_0^1 \phi(x) \, dF^{-1}(x) \leqslant \int_0^1 \phi(x) \, dG^{-1}(x), \qquad \forall \phi \in \mathcal{M}_n.$$

*Proof.* (If part.) By Lemma 1, $F \geqslant_n^{-1} G$ is equivalent to

$$\int_0^x (x - y)^{n-1} \, dF^{-1}(y) \geqslant \int_0^x (x - y)^{n-1} \, dG^{-1}(y)$$

which implies

$$\int_0^1 \int_0^x (x - y)^{n-1} \, dF^{-1}(y) \, d\tau(x) \geqslant \int_0^1 \int_0^x (x - y)^{n-1} \, dG^{-1}(y) \, d\tau(x).$$

By Fubini's theorem, this is equivalent to

$$\int_0^1 \phi(y)\, dF^{-1}(y) \leqslant \int_0^1 \phi(y)\, dG^{-1}(y),$$

where

$$\phi(x) = -\int_x^1 (s-x)^{n-1}\, d\tau(s).$$

(Only if part.) We have

$$F_n^{-1}(x) = \int_0^1 \phi_x(y)\, dF^{-1}(y),$$

where

$$\phi_x(y) = \frac{(x-y)^{n-1}\, \chi_{[0,x]}(y)}{(n-1)!} = \int_y^1 (z-y)^{n-1}\, d\tau_x(z)$$

and

$$\tau_x(y) = \frac{\chi_{[x,1]}(y)}{(n-1)!};$$

$\chi_A$ is the indicator of the set $A$. ∎

The theorem states that the class of inequality measures of the form

$$\int_0^1 \phi(x)\, dF^{-1}(x) \qquad \phi \in \mathcal{M}_n$$

is coherent with $\geqslant_n^{-1}$.

## 4. Conclusion

We have introduced a sequence of partial orders $\geqslant_n^{-1}$ for distributions which are based on income differentials and are progressively finer, in the sense that, if $n \geqslant m$, then $\geqslant_n^{-1}$ can order all the pairs of distributions that are ordered by $\geqslant_m^{-1}$ (and some more). Therefore, as we pass from $\geqslant_n^{-1}$ to $\geqslant_{n+1}^{-1}$, none of the previously performed comparisons is denied, and some others are added.

When $n=1$, the maximum incomes of the $(100\alpha)\%$ $(0 \leqslant \alpha \leqslant 1)$ poorest parts of the populations are compared. When $n=2$, an integration is performed, and the cumulated incomes of the $(100\alpha)\%$ $(0 \leqslant \alpha \leqslant 1)$ poorest parts of the populations are compared. When $n=3$, a further integration is performed, and so on.

As $n$ increases, the importance of the lower incomes is stressed more and more. In this respect, the orders $\geqslant_n^{-1}$ may be given an interpretation in terms of ethical flexibility, which is analogous to the one invoked by Donaldson and Weymark for their $S$-Ginis. Some differences are worth noticing. $S$-Ginis induce linear orders, and it may happen that two $S$-Ginis give contradictory results, namely $\Xi_k(F_X) > \Xi_k(F_Y)$ and $\Xi_h(F_X) < \Xi_h(F_Y)$, whereas two partial orders $\geqslant_k^{-1}$ and $\geqslant_h^{-1}$ can never give opposite results. As Proposition 1 states, each order $\geqslant_h^{-1}$ is a refinement of the previous ones. This fact also explains why, if we define an order $\geqslant_\infty^{-1}$, for which

$$F \geqslant_\infty^{-1} G \qquad \text{iff} \quad \exists n \in \mathcal{N} \text{ s.t. } F \geqslant_n^{-1} G,$$

then $\geqslant_\infty^{-1}$ is not a minmax order. An heuristic justification of this claim follows. A minmax order $\geqslant^{\text{minmax}}$ depends only on the sign of $F^{-1}(x) - G^{-1}(x)$ as soon as this function becomes different than zero. $F \geqslant^{\text{minmax}} G$ iff $\exists \varepsilon > 0$, $\exists \delta \geqslant 0$, such that

$$F^{-1}(x) - G^{-1}(x) = 0 \qquad \forall x \in [0, \delta)$$

and

$$F^{-1}(x) - G^{-1}(x) > 0 \qquad \forall x \in (\delta, \varepsilon).$$

It is clear that this condition can be satisfied even if $\int x\, dF < \int x\, dG$. But $\int x\, dF \geqslant \int x\, dG$ is a necessary condition for $F \geqslant_1 G$, and therefore for $F \geqslant_1^{-1} G$. Therefore, if $\geqslant_\infty^{-1}$ is a refinement of $\geqslant_1^{-1}$, then it cannot be minmax. Another consequence of the above claim is that $\geqslant_\infty^{-1}$ is not a linear order. A thorough discussion of the order $\geqslant_\infty$ (the analog of $\geqslant_\infty^{-1}$ for distribution functions) can be found in Fishburn [8, 9].

We have proved that $F \geqslant_{n+1}^{-1} G$ implies $\Xi_k(F) \geqslant \Xi_k(G)$ for all $k \geqslant n$. This result stems from a necessary condition for $\geqslant_{n+1}^{-1}$, which is expressed in terms of expectations of order statistics. We think that the result is interesting, since it gives an intuitive meaning to the partial order $\geqslant_n^{-1}$. $E(X_1 \wedge \cdots \wedge X_k) \geqslant E(Y_1 \wedge \cdots \wedge Y_k)$, $(k \geqslant n-1)$ is necessary for $F_X \geqslant_n^{-1} F_Y$, where $X_1, ..., X_k$ are i.i.d. from $F_X$ and $Y_1, ..., Y_k$ are i.i.d. from $F_Y$. If we indicate with $\hat{F}_{X_1, ..., X_k}$ the empirical d.f. of $X_1, ..., X_k$, then $(X_1 \wedge \cdots \wedge X_k) = \hat{F}^{-1}_{X_1, ..., X_k}(1/k)$. This shows that as $n$ increases lower incomes become more important and higher incomes less important for $\geqslant_n^{-1}$.

From the absolute ordering $\geqslant_n^{-1}$ it is possible to derive a relative ordering, based on income shares, simply by dividing the r.v.'s by their expectations. All the previous results can be rephrased in this new framework, and coherence of relative $S$-Ginis $I_k$ with $\geqslant_n^{-1}$ $(n \geqslant k-1)$ can be established. It is worth noting that when $n=2$ we have the Lorenz ordering, and when $n=1$ the relative order does not make any sense, since two distributions with the same mean either coincide or are noncomparable with $\geqslant_1$.

We have considered orders $\geqslant_n^{-1}$ only for $n$ an integer. By resorting to fractional integrals, we can generalize the entire argument to $\geqslant_\alpha^{-1}$ with $\alpha \geqslant 1$ real (see Fishburn [7, 8] for $\geqslant_\alpha$).

If the intention is to stress the importance of the richest part of the population, rather than the poorest, then it is possible to define a sequence of "illfare" stochastic dominances by iteratively integrating the survival function rather than the distribution function. A class of coherent measures may be obtained as well (see Donaldson and Weymark [4]).

As a final remark, we want to recall that the idea of using the inverse of a distribution function, rather than the distribution function itself, as a weighting measure has been used by Yaari [16] in the framework of theory of choice under risk. As Yaari himself mentions, the use of this new approach (called dual theory of choice under risk) solves a "paradox" due to Newbery [13] according to which no von Neumann–Morgenstern utility ranks distributions in the same order as their Gini indices.

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## References

1. A. B. Atkinson, On the measurement of inequality, *J. Econ. Theory* **2** (1970), 244–263.
2. H. A. David, "Order Statistics," 2nd ed., Wiley, New York, 1981.
3. J. Davies and M. Hoy, A counterexample to theorem 1 of "A note on stochastic dominance and inequality measures" by P. Muliere and M. Scarsini, mimeo.
4. D. Donaldson and J. A. Weymark, A single-parameter generalization of the Gini indices of inequality, *J. Econ. Theory* **22** (1980), 67–86.
5. D. Donaldson and J. A. Weymark, Ethically flexible Gini indices for income distribution in the continuum, *J. Econ. Theory* **29** (1983), 353–358.
6. R. Dorfman, A formula for the Gini coefficient, *Rev. Econ. Statist.* **41** (1979), 146–149.
7. P. C. Fishburn, Continua of stochastic dominance relations for bounded probability distributions, *J. Math. Econ.* **3** (1976), 295–311.
8. P. C. Fishburn, Continua of stochastic dominance relations for unbounded probability distributions, *J. Math. Econ.* **7** (1980), 271–285.
9. P. C. Fishburn, Stochastic dominance and moments of distributions, *Math. Oper. Res.* **5** (1980), 94–100.

10. J. FOSTER AND M.-C. NG, Remarks on "A note on stochastic dominance and inequality measures" by Muliere and Scarsini, mimeo.
11. J. L. GASTWIRTH, A general definition of the Lorenz curve, *Econometrica* **39** (1971), 1037–1039.
12. A. MARSHALL AND I. OLKIN, "Inequalities: Theory of Majorization and its Applications," Academic Press, New York, 1979.
13. D. NEWBERY, A theorem on the measurement of inequality, *J. Econ. Theory* **2** (1979), 264–266.
14. T. ROLSKI, Order relations in the set of probability distributions functions and their applications in queueing theory, *Dissertationes Math.* **132** (1979).
15. A. F. SHORROCKS AND J. E. FOSTER, Transer sensitive inequality measures, *Rev. Econ. Studies* **54** (1987), 485–497.
16. M. E. YAARI, The dual theory of choice under risk, *Econometrica* **55** (1987), 95–115.
17. S. YITZHAKI, Stochastic dominance, mean variance, and Gini's mean difference, *Amer. Econ. Rev.* **72** (1982), 178–185.
18. S. YITZHAKI, On an extention of the Gini inequality index, *Int. Econ. Rev.* **24** (1983), 617–628.
19. S. YITZHAKI AND I. OLKIN, Concentration curves, Department of Statistics, Stanford University, 1986.