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RESEARCH PAPER

Evaluating groups with the generalized Shapley value

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Abstract Following the original interpretation of the Shapley value as a priori evaluation of the prospects of a player in a multi-person interaction situation, we intend to apply the *Shapley generalized value* (introduced formally in Marichal et al. in Discrete Appl Math 155:26–43, 2007) as a tool for the assessment of a group of players that act as a unit in a coalitional game. We propose an alternative axiomatic characterization which does not use a direct formulation of the classical efficiency property. Relying on this valuation, we also analyze the profitability of a group. We motivate this use of the Shapley generalized value by means of two relevant applications in which it is used as an objective function by a decision maker who is trying to identify an optimal group of agents in a framework in which agents interact and the attained benefit can be modeled by means of a transferable utility game.

Keywords Game theory · TU games · Shapley value · Generalized values · Group values

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1 Introduction

In 1953, Lloyd Shapley proposed his revolutionary approach to the problem of allocation of resources in a situation of multiple interactions between agents that are willing to cooperate in different ways. His solution, known as the Shapley value, meant a cornerstone in the theory of cooperative TU games, and generated a vast literature in which the authors attempt to cover many situations when a good should be shared in a cooperative situation. Since then, a number of authors have modified and generalized the Shapley value in order to extend it for more general situations. In this framework, weighted Shapley values have appeared (Shapley 1953b; Kalai and Samet 1987), as well as the Myerson value for games with graph restricted communication (Myerson 1977), values for games with coalition structures (Aumann and Drèze 1974; Owen 1977; Hart and Kurz 1983), and the *generalized values* (Marichal 2000), which are in fact the main issue of this paper, among others.

In Marichal (2000), it is defined the concept of *generalized values* to measure the overall influence of every coalition in a game, proposing a theoretical framework that extends classical individual values to the case of coalitions. Then, in Marichal et al. (2007) the authors propose axiomatizations of two classes of generalized values: probabilistic generalized values and generalized semivalues. In this paper we focus on the power of their Shapley generalized value -which is a special case of probabilistic generalized value- as a priori evaluation of the prospects of a group of players in a multi-person game; the players are supposed to act as a group, and the valuation takes into account the power of groups in their various cooperation opportunities without imposing on the other agents any concrete coalition structure.

We think that the term "generalized" is somewhat ambiguous, as there are other definitions of generalized Shapley value in the literature (Hamlen et al. 1980; Gul 1989; McQuillin 2009). Because of this, and also because our study is focused on group evaluation, we will refer to this particular instance of generalized value as *Shapley group value* in most parts of our paper.

Recall that the original idea of Shapley was to use his value as a tool for evaluating agents immersed a cooperative game; we return to this framework, making use of the generalized Shapley value for evaluating *groups of agents* in the game. We describe the applicability of the value in three different settings which share two relevant features: (i) the objective is the selection of an optimal group, rather than the best individual; and (ii) the performance of a group depends on its interaction with the rest of agents. In this context, to maximize the characteristic function entails a too restrictive assumption over the rest of agent's behavior, i.e., only one scenario (mainly, the worst one) is evaluated. On the contrary, we show that maximizing the *value of a group* allows us to consider a more general setting, in which more than one scenario concerning other agents' actions is taken into account.

A key observation in our proposal is that we do not need to suppose necessarily that the players know each other nor agree to act jointly; instead, we assume the existence of an *external agent*, the decision maker, that is able to coordinate the actions of the members of the group. This is the case for instance of terrorist organizations like Al Qaeda, or secret societies in which there exists a leader (or a set of leaders) who sends a common signal that all the agents in the group are willing to follow. In this work we describe alternative situations in which this type of external coordination occurs.

Now we briefly describe the contents of our paper. Section 2 is devoted to a general presentation of the problem we deal with. We first introduce some standard concepts and notation on Game Theory that will be used throughout this paper, and then we describe two different cases in which the need for a group valuation arises. In Sect. 3 we recall the notion of *generalized value* (Marichal et al. 2007) and we carefully describe their *Shapley generalized value*, what which we call the *Shapley group value*. It is proposed an axiomatic characterization of the value, which is proved in "Appendix", and we also analyze an illustrative example in the framework of diffusion in social networks. In Sect. 4 we analyze the profitability of a group by means of comparing its valuation as a group with the sum of its members' individual Shapley values. In Sect. 5 we explore one of its potential applications previously considered in Sect. 2. Section 6 concludes the paper.

2 Motivation and notation

A cooperative game in coalitional form with side payments, or with transferable utility, is an ordered pair (N, v), where N is a finite set of players and $v : 2^N \to \mathbb{R}$, with $2^N = \{S \mid S \subset N\}$, is a *characteristic function* on N satisfying $v(\varnothing) = 0$. For any coalition $S \subset N$, $v(S) \in \mathbb{R}$ is the *worth* of coalition S and represents the reward that coalition S can achieve by itself if all its members act together. Since we will restrict to the case of TU games in the sequel, we will refer to them simply as *games*. For brevity, throughout the paper, the cardinality of sets (coalitions) N, S and C will be denoted by appropriate small letters n, s and c, respectively. Also, for notational convenience, we will write singleton $\{i\}$ as i, when no ambiguity appears.

Let $\mathcal{U} = \{1, 2, \ldots\}$ be the *universe of players*, and let $\mathcal{N}$ be the class of all non-empty finite subsets of $\mathcal{U}$. For an element N in $\mathcal{N}$, let $\mathcal{G}_N$ denote the set of all characteristic functions on player set N, and let $\mathcal{G} = \bigcup_{N \in \mathcal{N}} \mathcal{G}_N$ be the set of all characteristic functions.[1] A *value* φ is a map which associates to each game $v \in \mathcal{G}_N$, $N \in \mathcal{N}$, a real vector $\varphi(N, v) = (\varphi_i(N, v))_{i \in N} \in \mathbb{R}^N$, where $\varphi_i(N, v) \in \mathbb{R}$ represents the *value* of player i, $i \in N$. Shapley (1953a) defines his value as follows:

$$\phi_i(N, v) = \sum_{S \subset N \setminus i} \frac{s!(n-s-1)!}{n!} \big(v(S \cup \{i\}) - v(S)\big), \quad i \in N. \tag{1}$$

The *value* $\phi_i(N, v)$ of each player, which is a weighted average of his marginal contributions, admits different interpretations, such as the *payoff* that player i receives when the Shapley value is used to predict the allocation of resources in multiperson interactions, or his *power* when averages are used to aggregate the power of players in their

[1] We will use interchangeably the two terminologies, game and characteristic function, when no ambiguity appears.

various cooperation opportunities. As announced in the introduction, our proposal for measuring the value of a group is based on the question originally addressed by Shapley in his seminal paper: we interpret the value as the *expectations* of a player (group) in a game (N, v). In other words, $\phi_i(N, v) \in \mathbb{R}$ is an *a priori* value that measures the prospects of player $i \in N$ in the game $v \in \mathcal{G}_N$, and can be used as an objective function for selecting *key players*. Note that in general $v(i) \neq \phi_i(N, v)$, which is precisely the basis for not taking $v(S)$ as a valuation of the prospects of group S; in fact, this observation has motivated a series of papers on ranking, where the problem of how to extend a ranking over single objects to another ranking over all possible collections of objects -taking into account the fact the possible interactions between the grouped objects-, is addressed. See for instance Moretti and Tsoukias (2012), and Lucchetti et al. (2015).

The approach just discussed is undertaken in the next two cases, already considered in the literature:

(i) In Lindelauf et al. (2013), the authors introduce a game-theoretic approach to identify the key players in a terrorist network. They considered four different weighted extensions of the *connectivity game* (Amer and Gimenez 2004) to capture the structure of the terrorist organization as well as additional individual information about the terrorists, and then they proposed to calculate the Shapley value of each game in order to identify the key players. In Sect. 5, where we analyze in detail this application, we recover the formal definitions of those games.

(ii) In Narayanam and Narahari (2011), the authors also introduce a game-theoretic approach to address the *target set selection problem* in the framework of diffusion of information, when it is assumed that each agent has two possible states: *active*, if he has adopted the information that is being propagated, and *inactive* otherwise. The authors define a game (N, v), that taking into account the stochastic diffusion process, measures the expected number of active nodes at the end of the diffusion process when initially all agents in coalition S are active, whereas all agents in $N \backslash S$ are inactive. Then, they propose to calculate the Shapley value of the game in order to rank the agents. Taking into account that the k agents with highest Shapley value are not in general the optimal set of k agents, they propose an heuristic procedure, based on the Shapley value of each agent and the social network structure, to select the *key set* of k agents. We will explain this application in detail in next Sect. 3.

Note that in the examples considered above, there exists in fact an *external decision maker* who is interested in finding an optimal *group* of agents, rather that an optimal agent:

(i) In the first example the police wants to identify a small group of terrorists to neutralize in order to break up the criminal organization. Or, it could be the case, that they were interested in selecting a small group of criminals to mislead in order to optimally diffuse their own information through the network (by using them as seed). In the first case, the goal could be to find the group of terrorists of a given size $1 \leq k < n$ whose removing turns into a maximum reduction of the criminal activity. In the second case, the goal could be to find a group of minimum size that achieves a given percentage of information spreading. In both cases, the police needs an *objective function* which evaluates the a priori performance of every group in each of the two settings.

(i) In the *target set selection problem* of example two, the goal is to find a set of k key agents that would maximize the spreading of information through the network. Thus, we need again an objective function which measures the a priori ability of each group to spread information.

It must also be remarked that, analogously to what happens for the individual case, in which $v(i)$ is not a proper valuation of the performance of player $i \in N$ in the game, a direct use of $v(S)$ to measure the a priori value of group S is not in general the best approach to solve this problem. For instance, to maximize $v(\cdot)$ in the second example implies a pessimistic scenario in which none of the agents out of coalition S whose diffusion power is being evaluated adopts the product spontaneously. The same argument remains valid for the other situation considered above. Thus, since measuring the expected value of a group is a relevant question, and taking into account that the k most valuable agents (from the individual point of view) do not form in general the most valuable group of k agents, the need for a specific group valuation is clear.

3 The generalized Shapley value: a tool for evaluating groups

A good choice for the valuation of groups turns out to be the *generalized Shapley value*, defined by Marichal et al. (2007). In this section, taking into account that it is not the main goal of a group valuation to distribute a fixed amount, we propose an alternative set of axioms for it which does not include a direct formulation of the classical efficiency axiom. We end this section with an example in which the *a priori* power of diffusion of a group of agents in a social network is assessed by means of the generalized Shapley value.

First, in order to consider an accurate valuation we must consider what group integration means for the applications we have in mind. In this framework, group integration does not necessarily imply that agents in C make an agreement to act jointly. For instance, going back to the diffusion of information case, there exists a external agent who can activate the nodes that are used as seeds to diffuse the innovation through the network, and the activated nodes are not in general aware about the other selected seeds' identities. The same occurs when the police selects a group of terrorists to turn back into double agents, or to misinform in order to spread their misinformation through the criminal organization network. Therefore, when measuring group C's expectations we will evaluate them like a *unit* anyway, and we will adopt the *merging of players* approach of Derks and Tijs (2000), who analyze the *profitability* of group formation in a more general setting[2] by means of considering Lehrer's (1988) type of merging. In that case all the agents of C are replaced by a single player $\mathbf{c}$, who can act as a proxy of any agent in C. In this setting, the generalization of the Shapley value to groups turns out to be the *Shapley generalized value* (Marichal 2000; Marichal et al. 2007). As remarked in the introduction, we prefer the name "Shapley group value"

[2] But not, as said above, the problem of assigning values to groups.

because our work is more aimed to evaluate prospects than to generalize individual values.

Formally, let $v \in \mathcal{G}_N$, and let $C \subset N \in \mathcal{N}$ be any non-empty coalition. Now, let us consider the so called *C-partition*, denoted by $\mathcal{P}_C$, consisting of the compartments C and the one-person coalitions of players outside C. Then, the *merging game*—in the sense of Lehrer (1988)—with respect to $\mathcal{P}_C$ is the $(n - c + 1)$-person cooperative game (N_C, v_C), where the agent set $N_C = (N \backslash C) \cup \{\mathbf{c}\}$ with $\mathbf{c}$ as a single proxy player $\mathbf{c} \equiv C$, and v_C is of the form:

$$v_C(S) = \begin{cases} v(S), & \text{if } \mathbf{c} \notin S, \\ v(S \cup C) & \text{if } \mathbf{c} \in S, \end{cases} \quad \forall\, S \subset N_C. \tag{2}$$

Definition 1 (Marichal et al. 2007) A *generalized value* is regarded as a *valuation* mapping ξ^g that assigns for every game $v \in \mathcal{G}_N$ and every $C \subset N$ a real number $\xi^g(C; N, v) \in \mathbb{R}$ that reflects the power of coalition C in the game v, and such that $\xi^g(\varnothing; N, v) = 0$.

Remark 1 In the context of this paper, generalized values will be mainly considered as tools for evaluating groups. Hence, we will call them "group values".

Definition 2 (Marichal et al. 2007) The *Shapley group value* is the group value that assigns for every $v \in \mathcal{G}_N$, $N \in \mathcal{N}$, the valuation mapping $\phi^g(\cdot; N, v)$ given by:

$$\phi^g(C; N, v) = \phi_{\mathbf{c}}(N_C, v_C), \text{ for each group } \varnothing \neq C \subset N.$$

Observe that this definition coincides with the definition of generalized Shapley value of Marichal et al. (2007), although these authors introduced it in a different way and only obtain the value in the merging game as a consequence of the reduced partnership axiom. Since each $\phi^g(C; N, v)$ is obtained by applying the Shapley value to a merging game, it is remarkable that for every coalition with at least two players the corresponding merging game is different. One-person coalitions $C_1 = \{i\}$ and $C_2 = \{j\}$ are the unique cases in which the two merging games, (N_{C_1}, v_{C_1}) and (N_{C_2}, v_{C_2}), are the same for two different groups C_1 and C_2. Trivially these two merging games coincide with (N, v).

Note that the previous merging game is also called the *quotient game* (Owen 1977) of (N, v) with respect to the *coalition structure*, or in other terms to the *system of a priori unions*, on $N \subset \mathbb{N}$ given by the partition $\mathcal{P}_C = \{C, \{j\}, j \notin C\}$ of N. Moreover, the Shapley group value of $C \subseteq N$ in the game (N, v) equals the a priori expectation of group C in the first step of the process described in Owen (1977). However, as the author points out, the principal problem solved in that paper "lies in determining a division of that total amount among the several member of the union C", not to explore the prospects of the group C itself.

We are mainly interested in analyzing properties of the Shapley group value which are relevant from the point of view of group valuation and its applications.

We first recall some definitions. A game (N, v) is a *unanimity game* if there exists a coalition $S \subset N$ such that for every $T \subset N$, $v(T) = 1$ if $S \subset T$, and $v(T) = 0$

otherwise. In this case, the game is usually denoted by (N, u_S). Unanimity games are a basis of the vector space $\mathcal{G}_N$ of all games with the player set N. Also recall that given $v \in \mathcal{G}_N$, $i \in N$ is a *dummy player* if $v(S \cup i) = v(S) + v(i)$ for all $S \subset N$. A dummy player with $v(i) = 0$ is said to be a *null player* in v. A game $v \in \mathcal{G}_N$ is *monotonic* if $v(T) \geq v(S)$ for all $S \subset T \subset N$.

Here are the properties we need:

Properties

Let ξ^g be a real mapping defined over the set of all games $\mathcal{G}$. Then, ξ^g verifies:

(P1) *G-null player*, if $\xi^g(C \cup i; N, v) = \xi^g(C; N, v)$ for all $C \subsetneq N \in \mathcal{N}$, $i \in N \backslash C$ and $v \in \mathcal{G}_N$, whenever i is a null player;

(P2) *G-linearity*, if $\xi^g(C; N, \alpha_1 v + \alpha_2 w) = \alpha_1 \xi^g(C; N, v) + \alpha_2 \xi^g(C; N, w)$ for all $C \subset N \in \mathcal{N}$, $\alpha_1, \alpha_2 \in \mathbb{R}$, and games $v, w \in \mathcal{G}_N$, where $\alpha_1 v + \alpha_2 w \in \mathcal{G}_N$ is given by $(\alpha_1 v + \alpha_2 w)(S) = \alpha_1 v(S) + \alpha_2 w(S)$ for all $S \subset N$;

(P3) *G-coalitional balanced contributions* (or G-CBC for short), if for all $C \subsetneq N \in \mathcal{N}$, $i, j \in N \backslash C$ and $v \in \mathcal{G}_N$, we have

$$\begin{aligned}(\xi^g(C \cup i; N, v) - \xi^g(C; N, v)) - (\xi^g(C \cup i; N \backslash j, v_{-j}) - \xi^g(C; N \backslash j, v_{-j}))\\ = (\xi^g(C \cup j; N, v) - \xi^g(C; N, v)) - (\xi^g(C \cup j; N \backslash i, v_{-i}) - \xi^g(C; N \backslash i, v_{-i})), \quad (3)\end{aligned}$$

where $v_{-i} \in \mathcal{G}_{N \backslash i}$ stands for the restriction of the characteristic function v to the set of players $N \backslash i$;

(P4) *G-symmetry over pure bargaining games* (or G-SPB for short), if $\xi^g(C; N, u_N) = \frac{1}{n-c+1}$ for each non-empty $C \subset N \in \mathcal{N}$, where (N, u_N) is the unanimity game with respect to the grand coalition.

G-linearity coincides with the *linearity* (L) axiom considered in Marichal et al. (2007), while G-null player is equivalent to the *First dummy coalition axiom* (DC') that appears in Marichal et al. (2007) since we deal with finite games.

G-coalitional balanced contributions generalizes the balanced contribution property which adding efficiency characterizes the Shapley value (Myerson 1977). G-CBC states that for any group $C \subset N \backslash \{i, j\}$, the impact of player j's presence over the marginal contribution of player i to the value of group C equals the impact of player i's presence over the marginal contribution of player j to the value of the same group C. This idea generalizes the original balanced contribution property of Myerson: "… *that player i contribute to player j's payoff what player j contributes to player i's payoff.*" (Winter 2002). Note that in this case payoffs refer to players' values, and therefore the original property is a particular case of G-CBC for $C = \varnothing$.

G-SPB leads to regard each group as one representative, independent of the number of original players it is composed of, when all players are strictly necessary. In a voting game in which the vote of all players is needed in order to pass a bill, all of them are equally powerful regardless of the weights they originally have.

Theorem 1 *The unique group value over the set of all games $\mathcal{G}$ verifying G-null player, G-linearity, G-CBC, and G-SPB is the Shapley group value ϕ^g.*

Not to interrupt the natural flow of our arguments, and taking into account that we are mainly interested in the applications, we postpone the proof of this theorem to "Appendix". We also check there that all the considered properties are necessary to guarantee the uniqueness of the Shapley group value ϕ^g.

Remark 2 As we commented before, G-SPB leads players to act as one representative in pure bargaining games when all players are strictly necessary, and this combined with G-null player and G-CBC, lead to the same behavior in unanimity games (N, u_S), $S \subset N$. In fact, assume the following:

$$\xi^g(C; N, u_N) = \frac{c}{n}, \text{ for every non-empty group } C \subset N \in \mathcal{N}. \tag{4}$$

Then, condition (4), G-null player, G-linearity and G-CBC characterize the additive group value $\mathcal{A}\phi^g$ defined as the sum of the individual Shapley values of the involved players, where $\mathcal{A}\phi^g(C; N, v) := \sum_{i \in C} \phi_i(N, v)$, for every $C \subset N$ and for all $N \in \mathcal{N}$.

We conclude from the previous remark that in case we want to use a group valuation with the Shapley value standards that accounts for the synergy among group members, then we must use the Shapley group. Next we illustrate this fact by means of the diffusion game defined by Narayanam and Narahari (2011). This example will help to evaluate the advantage of using the Shapley group value instead of only taking into account the information provided by the Shapley individual value, or following a generalist strategy (as Narayanam and Narahari), that considers the network information always in the same manner and only uses local information, i.e. if two agents are directly connected or not.

Example: target set in a diffusion problem

Let us recall first the *target set selection problem* in the framework of diffusion of information. Let $N = \{1, 2, \ldots, n\}$ be a finite and fixed set of agents who interact according to a social network. Relations between agents are exogenously given by a weight matrix $\boldsymbol{W}$ whose entries are understood as *influence weights*; in particular, w_{ij} quantifies the weight that agent i assigns to agent j. It is assumed that these weights are normalized in such a way that $\sum_{j \in N_i} w_{ij} \leq 1$, where N_i represents the set of neighbors of agent i (that is, $N_i := \{j \in N \mid w_{ij} > 0\}$, for all $i \in N$).

It is also assumed that each agent has two possible states: *active*, if he has adopted the behavior or innovation that is being propagated, and *inactive* otherwise, characteristics indicated by 1 or 0, respectively. From a dynamic point of view, it is assumed that the status of the agents may change as time goes by. At each date, agents communicate with their neighbors in the social network and update their state. Many different updating process have been proposed. Narayanam and Narahari (2011) consider the *linear threshold model* (Granovetter 1978) as the updating process: all agents that were active in step $(t-1)$ remain active at step t; and every inactive agent at step $(t-1)$ becomes active if the sum of the weights of his active neighbors' from the previous period is at least θ_i, a threshold which represents the weighted fraction of the neighbors of i that must become active in order to activate agent i. Formally, a *state* of the social network

$(N, \boldsymbol{W})$ is a tuple $\boldsymbol{x} \in \{0, 1\}^n$, $x_i \in \{0, 1\}$ being the state of agent i, $i = 1, \ldots, n$, where n is the cardinality of N. Thus, the updating process considered by Narayanam and Narahari (2011) is given by:

$$x_i^t = \begin{cases} 1, & \text{if } x_i^{t-1} = 1 \text{ or } \sum_{j \in N_i} w_{ij} x_j^{t-1} > \theta_i, \\ 0, & \text{otherwise.} \end{cases}, \quad \text{for all } t \geq 1.$$

In this conditions, the stochastic process that describes the diffusion of information assumes that all thresholds θ_i, $i \in N$, are chosen uniformly at random from the interval $[0, 1]$ initially. Then, let $\hat{\sigma}(S)$ be the the expected number of active nodes (or adopters) at the end of the diffusion process when initially all agents in S are active, whereas all agents in $N \backslash S$ are inactive (i.e. $x_i^0 = 1$, for all $i \in S$ and $x_i^0 = 0$, for all $i \in N \backslash S$).

The *target set selection problem* consists on selecting the *key set* of k agents that maximizes the expected number of adopters. Kempe et al. (2005) propose a *greedy* heuristic to select the key group of a given size k based on the objective function $\hat{\sigma}(\cdot)$. As we have remarked previously (see page 4), this approach assumes a pessimistic scenario in which none of the agents out the key group whose diffusion power is being evaluated adopts the behavior spontaneously. Taking this consideration into account, Narayanam and Narahari (2011) propose to define a game (N, v), being $v(S) := \hat{\sigma}(S)$, for all non-empty coalition $S \subseteq N$, and being $v(\varnothing) := 0$, and calculate the Shapley value of the game in order to rank the agents. Taking into account that the k agents with highest Shapley value are not in general the optimal set of k agents, they propose an heuristic procedure based on the Shapley value of each agent and the social network structure, to select the *key set* of k agents, which they called SPIN. The agents are selected according to their individual Shapley values as long as they are not connected to any of the already selected ones. If there are no remaining nodes with this property and the size of the selected group is smaller than k, then the process selects among the remaining ones according to their ranking. The authors propose the following example depicted in Fig. 1 to illustrate the fact that the k agents with highest Shapley value are not in general the optimal set of k agents, and to motivate their proposal. We will use the same example to show the ability of the Group Shapley Value for capturing the externalities that emerge from the network structure. Thus, using the Group Shapley value as a valuation function for a group is a better option than following a general heuristic approach regardless of the specific structure of the network.

In this case, the weight matrix $\boldsymbol{W}$ derived from the graph (N, Γ) is given by $w_{ij} = \frac{1}{d_i}$ for all j such that $\{i, j\} \in \Gamma$, and being d_i the degree of node i, i.e. $d_i = |\{j \in N | \{i, j\} \in \Gamma\}|$. The weight $w_{ij} = 0$ for all non directly connected agents.

In next table the results obtained for $k = 1, 2, 3, 4$ from the greedy heuristics (Kempe et al. 2005) and SPIN (Narayanam and Narahari 2011) are shown, joint with the point estimation of the Shapley value of every group[3] and a 95% confidence

[3] They have been estimated by means of a Monte Carlo simulation following Castro et al. (2009) of the value of every group obtained with 7000 replications of the experiment.

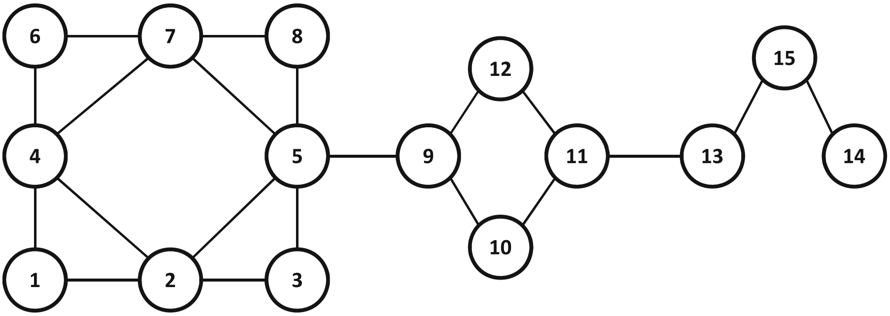

Fig. 1 Narayanam and Narahari (2011) example

Table 1 Shapley group valuations and estimated diffusion. Narayanam and Narahari (2011) example

k	SPIN S_k^{SP}	$\hat{\sigma}(S_k^{SP})$	$\hat{\phi}^g(S_k^{SP})$	95% CI	Greedy S_k^{Gr}	$\hat{\sigma}(S_k^{Gr})$	$\hat{\phi}^g(S_k^{Gr})$	95% CI
1	{5}	4	1, 43	(1, 39; 1, 47)	{5}	4	1, 43	(1, 39; 1, 47)
2	{5, 4}	7	2, 39	(2, 33; 2, 45)	{5, 11}	8	2, 52	(2, 46; 2, 58)
3	{5, 4, 11}	10	3, 55	(3, 47; 3, 63)	{5, 11, 2}	10	3,37	(3, 30; 3, 44)
4	{5, 4, 11, 15}	12	4, 57	(4, 46; 4, 67)	{5, 11, 2, 15}	12	4, 32	(4, 22; 4, 41)

interval. The rank list based on the individual Shapley value obtained by (Narayanam and Narahari 2011) is

$$RankList[] = \{5, 4, 2, 7, 11, 15, 9, 13, 12, 10, 6, 14, 3, 1, 8\}.$$

All the columns but the ones that make reference to the Shapley group value have been taken from the original paper from Narayanam and Narahari (Table 1).

The SPIN strategy for groups of size two selects agents 5 and 4, which lie in the same cluster, while the greedy strategy based on $\sigma(\cdot)$ selects agent 5 in the cluster and agent 11 in the queue; both agents have more diffusion capacity in the pessimistic scenario: $\hat{\sigma}(\{5, 4\}) = 7 < 8 = \hat{\sigma}(\{5, 11\})$, and also in average, the Shapley group value of {5, 11} is also higher. That is, the greedy strategy selects a better group than SPIN strategy does for $k = 2$. This is not the case for bigger groups.

For $k = 3$ and 4, although the selected groups are indistiguishable according to their estimated expected diffusion $\hat{\sigma}(\cdot)$ -that only takes into consideration the worst scenario in which no agent out of the targeted group C adopts the new behaviour spontaneously-, the groups selected by the SPIN strategy have bigger group Shapley values than those groups selected by the greedy strategy based on $\sigma(\cdot)$. Note that the Shapley group value of a group C evaluates its marginal contribution in all possible scenarios, in which every possible group of agents $S \subseteq N \backslash C$ out of group C whose diffusion power is being evaluated adopts also the new behaviour spontaneously. Therefore, it seems that SPIN strategy, which is also based on the Shapley value and thus takes into account also all possible scenarios, outperforms the simple greedy approach for groups with $k = 3$ and 4 agents.

However, the SPIN strategy heuristics is a very generalist strategy that takes into account the information provided by the network every time in the same way, and that only uses very local information; hence, good results cannot be always expected, as adjacent nodes are not redundant in general. On the contrary, the Shapley group value is able to identify sinergies and redundancies, no matter the network structure.[4] Thus, adopting a more sophisticated greedy approach based on the Shapley group value could be a better option that outperforms both strategies regardless of the size of the target group. Note that in this example, if we would use a greedy strategy based on the Shapley group value, the selected target groups would be $\{5\}$, $\{5, 11\}$, $\{5, 11, 4\}$ and $\{5, 11, 4, 15\}$, for $k = 1, 2, 3$ and 4, respectively. That is, we would select the best ones among those selected by SPIN algorithm and a Greedy heuristic based on the expected diffusion.

4 Profitability of a group

In Sect. 3 we have proposed to use the Shapley generalized value as a priori value of a group and we have given an alternative characterization. In this section we rely on this valuation to analyze the *profitability* of a group.

Next proposition shows that supperadditivity implies that the expected value of group C is at least the value that their members can assure for themselves. Moreover, if the game is monotonic larger groups are more valuable.

We also analyze when the integration of group C is *mergeable* in the sense of Derks and Tijs (2000). We end up this section studying the marginal effect that the incorporation of a new member has over an already integrated group. Theorem 2 relates this marginal effect with a measure of average complementarity between the entrant player and the incumbent group.

Proposition 1 *Let $N \in \mathcal{N}$ be any finite set of players, and v be any game in $\mathcal{G}_N$. Then, the Shapley group value ϕ^g verifies the following properties:*

(i) Group Rationality*: $\phi^g(C; N, v) \geq v(C)$ for every $C \subset N \in \mathcal{N}$ if the game $v \in \mathcal{G}_N$ is superadditive, and*
(ii) Monotonicity*: $\phi^g(C; N, v) \leq \phi^g(D; N, v)$ for every pair of coalitions $C \subset D \subset N \in \mathcal{N}$ if the game $v \in \mathcal{G}_N$ is monotonic.*

Now we will examine the profitability of the integration of group C measured as the difference between the Shapley value of group C, which represents its a priori valuation when they act as one representative, and the sum of the individual Shapley values of the involved players (the additive Shapley group value of C), i.e.

$$\phi^g(C; N, v) - \sum_{i \in C} \phi_i(N, v).$$

Profitability is analyzed by Derks and Tijs (2000), and also by Segal (2003). Recall that a coalition C is called *profitable* if the previous condition $\phi^g(C; N, v) -$

[4] This fact will be analyzed in next section.

$\sum_{i \in C} \phi_i(N, v)$ holds for C. Moreover, the *Harsanyi dividend* of a coalition C is defined as the number $d(C) = \sum_{K \subseteq C} (-1)^{|C|-|K|} v(K)$.

Now we can review the general results obtained by Derks and Tijs in this context.

Proposition 2 (Derks and Tijs 2000) *Let $N \in \mathcal{N}$ be any finite set of players, and v be any game in $\mathcal{G}_N$. Then, coalition $C \subset N \in \mathcal{N}$ is* profitable.[5] *whenever all coalitions with positive Harsanyi dividend are either contained in C or have at most one player in common with C.*

Derks and Tijs also propose some interesting types of games for which every coalition is profitable, or profitability can be guaranteed for certain kinds of coalitions.

The results of Segal (2003) rely on the *second-order difference operator* for a pair of players $i, j \in N$, and the *third-order difference operator* for three players $i, j, k \in N$.

The *second-order difference operator* for a pair of players $i, j \in N$ is defined as a composition of marginal contribution operators (i.e., first-order difference operators) as follows:

$$\Delta^2_{ij}(S; N, v) = v(S \cup \{i, j\}) - v(S \cup j) - v(S \cup i) + v(S), \quad \forall\, S \subset N \backslash \{i, j\}.$$

Here $\Delta^2_{ij}(S; N, v)$ expresses player i's effect over the marginal contribution of player j (or vice versa). Note that $v(S \cup \{i, j\}) - v(S) = \Delta^2_{ij}(S; N, v) + \Delta_i(S; N, v) + \Delta_j(S; N, v)$, and thus $\Delta^2_{ij}(S; N, v) > 0$ implies that the marginal contribution of players i, j as a group exceeds the sum of the individual marginal contributions of each player.

In fact, following Bulow et al. (1985), players i and j are said to be *complements* whenever $\Delta^2_{ij}(S; N, v) \geq 0$, for all $S \subset N \backslash \{i, j\}$. They are said to be *substitutes* whenever $\Delta^2_{ij}(S; N, v) \leq 0$, for all $S \subset N \backslash \{i, j\}$. Therefore, $\Delta^2_{ij}(S; N, v)$ can be interpreted as a measure of players i and j *interaction* with respect to the players in S.

Analogously, the *third-order difference operator* for players $i, j, k \in N$ is defined as $\Delta^3_{ijk}(\cdot; N, v) = \Delta_i(\Delta^2_{jk}(\cdot; N, v))$, for all $S \subset N \backslash \{i, j, k\}$. Here $\Delta^3_{ijk}(S; N, v)$ expresses player k's effect over the complementarity between players i and j with respect to the players in S. Again, the operator does not depend on the order of taking differences. The notions of second and third order difference operators have also been defined in Fujimoto et al. (2006).

Segal (2003) obtains the following result about profitability of groups of two players,[6] showing that the merging of two players $i, j \in N$ is profitable (unprofitable) whenever the presence of the outside players reduces (increases) the complementarity between the colluding players. This author also takes into account the profitability of the incorporation of a new member $j \in N \backslash C$ to an already integrated group C (point (ii) in next Proposition 3). In this case, profitability is measured with respect to the situation in which the players of group C are colluding. That is, profitability means

[5] Actually, Derks and Tijs (2000) refer to profitability as *mergeability*.

[6] Note that the kind of group integration we work with is equivalent to the *collusive* contracts considered by Segal (2003).

$$\phi^g(C \cup j; N, v) \geq \phi^g(C; N, v) + \phi_j(N_C, v_C). \tag{5}$$

Proposition 3 (Segal 2003) *Let $N \in \mathcal{N}$ be any finite set of players, and v be any game in $\mathcal{G}_N$. Then:*

- (i) *A coalition $C = \{i, j\} \subset N$ of two players is* profitable *(*unprofitable*) if $\Delta^3_{ijk}(S; N, v) \leq (\geq)\ 0$, for every coalition $S \subset N\backslash\{i, j, k\}$, and for all $k \in N\backslash C$. If the reverse inequalities hold, then group C is unprofitable.*
- (ii) *The union of the integrated group $C \subset N$ and player $j \notin C$ is profitable (unprofitable) if $\Delta^3_{ijk}(S; N, v) \leq (\geq)\ 0$, for every coalition $S \subset N\backslash\{i, j, k\}$, and for all $i \in C$, $k \in N\backslash(C \cup i)$.*

Condition (i) above states that profitability of the merging of players i and j is not directly related with their own complementarity. In fact, if we analyze the *games with indispensable players* application of Segal (2003) that models for instance the case of a firm that is indispensable to its workers, it holds that the union of two substitutable workers is profitable if their bargaining opponents (the remaining workers) enhances their substitutability. With respect to the union of the firm p and one of the workers d, which are always complements, this union is profitable (unprofitable) if worker d is a substitute (complement) of the rest of workers.

According to Segal's results it is clear that complementarity and substitutability are not directly related to profitability. However, Theorem 2 below shows how the Shapley group value incorporates those kind of relations among the players when evaluating the value of a group. The marginal contribution of a new entrant j to the incumbent group C is the sum of the a priori value of player j which does not depend on C, and the *average complementarity* between j and C. Formally:

Definition 3 Let $N \in \mathcal{N}$ be any finite set of players, and v be any game in $\mathcal{G}_N$, the *average complementarity* of players $i, j \in N$ is defined as the following average of second-order differences:

$$\psi_{ij}(N, v) := \sum_{S \subset N\backslash\{i,j\}} \frac{s!(n-s-1)!}{n!} \Delta^2_{ij}(S; N, v), \quad \text{for all } i \neq j \in N. \tag{6}$$

The average $\psi_{ij}(N, v)$ is taken over all the possible orders of $N = \{1, \ldots, n\}$, when all orders are considered equally probable. The second-order difference $\Delta^2_{ij}(S; N, v)$ is considered in all orders in which coalition S contains all players arriving between i and j, and i comes before j. It can be interpreted as an *interaction index* in the sense of Grabisch and Roubens (1999).

Theorem 2 *Let $N \in \mathcal{N}$ be any finite set of players, and v be any game in $\mathcal{G}_N$. Let $C \subset N$ be any group in N, and let $i \notin C$. Then, the* marginal contribution *of player $i \in N\backslash C$ to the Shapley group value of C equals:*

$$MC^g_i(C; N, v) := \phi^g(C \cup i; N, v) - \phi^g(C; N, v) = \phi_i(N\backslash C, v|_{N\backslash C}) + \psi_{\mathbf{c}i}(N_C, v_C). \tag{7}$$

The previous result shows that the value of a group results from a complex combination of independence and complementarity among its members. In particular, it is clear that the most valuable k agents from an individual point of view do not form in general the most valuable group of k agents. Let us illustrate this fact with the following simple example. Again we postpone the proof to "Appendix".

Example 1 Let us consider the following social network represented in Fig. 2 as an *undirected graph* (N, Γ), and the *connectivity game* (Amer and Gimenez 2004), which is defined as

$$v(S) = \begin{cases} 1, & \text{if } S \text{ is connected in } \Gamma \text{ and } |S| > 1, \\ 0, & \text{otherwise,} \end{cases} \quad \text{for all } S \subset N.$$

Here, $S \subset N$ is a *connected* coalition in (N, Γ) if for every two players $i \neq j$ in S, $\{i, j\} \in \Gamma$, or there exists a *path* between them which consists of nodes in S. That is, there exists a sequence of nodes and edges $\pi(i, j) = \{i = i_1, i_2, \ldots, i_{k-1}, i_k = j\}$, with $k \geq 2$ satisfying the property that for all $1 \leq r \leq k-1$, $\{i_r, i_{r+1}\} \in \Gamma$, and $i_r \in S$, for all $2 \leq r \leq k-1$.

In that case, the two most valuable players, according to their individual Shapley values are the two centers of the satellite stars, players 4 and 6. $\phi_i(N, v) = -\frac{8}{360}$, for all the leaves $i = 1, 2, 3, 7, 8, 9$, $\phi_4(N, v) = \phi_6(N, v) = \frac{139}{360}$, for the two centers, and $\phi_5(N, v) = \frac{130}{360}$ for the hub which intermediates between players 4 and 6. However, the most valuable group of two agents is the one composed by the hub and one out of the two centers. In fact,

$$\begin{aligned} \phi^g(\{4, 6\}; N, v) &= \phi_4(N, v) + \phi_6(N \backslash 4, v|(N \backslash 4)) + \psi_{46}(N, v) \\ &= \frac{139}{360} + \frac{1}{8} - \frac{1}{90} = \frac{1}{2}, \\ \phi^g(\{4, 5\}; N, v) &= \phi_4(N, v) + \phi_5(N \backslash 4, v|(N \backslash 4)) + \psi_{45}(N, v) \\ &= \frac{139}{360} + \frac{1}{56} + \frac{19}{72} = \frac{1}{2} + \frac{47}{280}. \end{aligned}$$

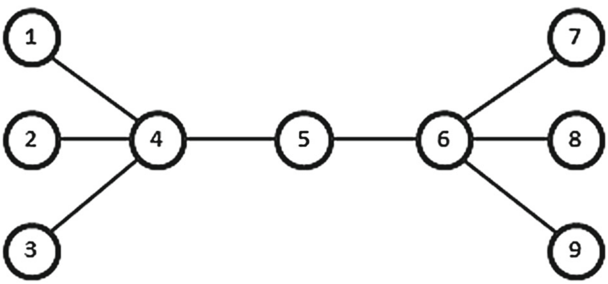

Fig. 2 Social network (N, Γ)

The result of this computation can be interpreted in the following lines: although the power of player 5 depends more on the presence of player 4 than the power of 6, the average complementarity of players 4 and 5 is greater than the average complementarity of players 4 and 6. □

5 Application: detecting a target group in terrorist networks

We will illustrate now the application of the Shapley group value to the two terrorist networks which have been considered by Lindelauf et al. (2013): the operational network of Jemaah Islamiyah's Bali bombing and the network of hijackers of Al Qaeda's 9/11 attack. We will also rely on Theorem 2 to estimate the level of interaction between a given incumbent group and a new potential entrant.

We want to emphasize that the goal of our study is to analyze which could be the advantages of using Shapley group value with respect to the use of individual value. This analysis should *not* be understood as predictive, and we are not interested in giving a new description or interpretation of an event for which it already exists a huge amount of literature. Rather we use this well-known network -whose complete structure has only been completely described after the attack- to test our valuation measures.

For the first case, Jemaah Islamiyah's Bali bombing attack, the mentioned authors use the game (N, v^{wconn}). Let (N, Γ) be the *undirected graph* which represents the terrorist network. The nodes in the finite set $N = \{1, \ldots, n\}$ are the terrorists, whereas the *edges* -i.e., unordered pairs of distinct nodes- represent the known relationships between the terrorists. In Fig. 3 the terrorist network we work with is represented.

Then, Lindelauf et al. (2013) define the game (N, v^{wconn}), which extends the connectivity game of Amer and Gimenez (2004) using information about relationships. In that game, a coalition must be connected in order to achieve a non-zero value. That is, players in coalition S must rely only upon their own connections in order to communicate among themselves. Then, since a terrorist cell tries to prevent discovery during the planning and execution phase of an attack, and taking into account the available data about the existing relationships,[7] the authors define the power of a coalition as the total number of relationships that exist within that coalition divided by the sum of the weights (representing frequency and duration of interaction) on those relationships;

$$v^{wconn}(S) = \begin{cases} \dfrac{\sum_{\substack{i,j \in S \\ i \neq j}} I_{ij}}{\sum_{\substack{i,j \in S \\ i \neq j}} f_{ij}}, & \text{if } S \text{ is connected in } \Gamma \text{ and } |S| > 1, \\ 0, & \text{otherwise,} \end{cases}, \quad \text{for all } S \subset N, \tag{8}$$

where f_{ij} is the weight assigned to relation $\{i, j\} \in \Gamma$ in the terrorist network, $I_{ij} = 1$, for every edge $\{i, j\}$ in Γ, and 0 otherwise.

[7] The authors collected the strength of existing relationships from Koschade (2006).

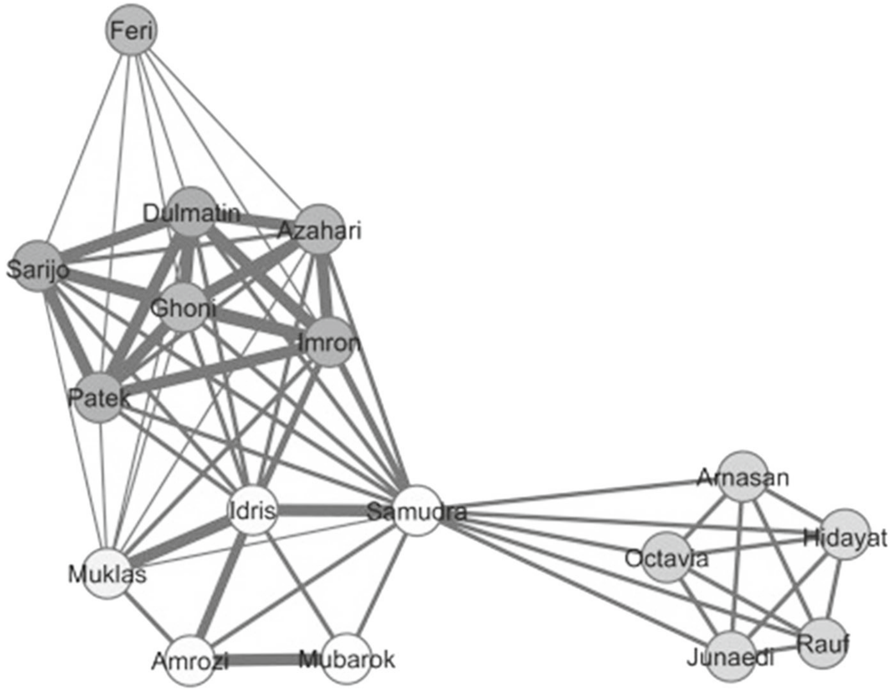

Fig. 3 Operational network of JI's Bali attack. Image taken from Lindelauf et al. (2013)

We obtain the following results concerning groups from one to four individuals. Following Castro et al. (2009), and taking into account that the marginal contributions in the extended connectivity games are computable in polynomial time, we have estimated with Monte Carlo simulation the Shapley group value of the examples, also in polynomial time. The results obtained are represented in Table 2, which includes the records for the best groups arranged in decreasing order of importance.

According to the individual rankings for the JI network based on the Shapley value, the five most valuable terrorists were, in decreasing order of importance: Samudra, Muklas, Feri, Azahari and Sarijo.

With respect to groups of two terrorists, the most valuable group is that composed by the two most important agents, {Samudra, Muklas}. However, the second group of size two in importance is {Samudra, Azahari}, improving the Shapley group value of {Samudra, Feri}, which equals 0.350, and takes the 15th place. In fact, Samudra has all direct contacts Feri has, and therefore Feri's presence in a group is somehow redundant if Samudra is already in it (see the estimated interactions below). According to what it is known about the attack, "*Samudra, an engineering graduate, played a key role in the bombings*", whereas Azahari is the bomb expert who was considered the "brain" behind the entire operation.

Table 2 Operational network of JI's Bali attack rankings

Individuals	Two agents	Three agents	Four agents
Samudra, 0.358	{Samudra, Muklas}, 0.453	{Samudra, Muklas, Azahari}, 0.442	{Samudra, Muklas, Feri, Azahari}, 0.466
Muklas, 0.048	{Samudra, Azahari}, 0.392	{Samudra, Muklas, Sarijo}, 0.435	{Samudra, Muklas, Feri, Sarijo}, 0.460
Feri, 0.032	{Samudra, Sarijo}, 0.386	{Samudra, Muklas, Patek}, 0.435	{Samudra, Muklas, Feri, Patek}, 0.460
Azahari, 0.012	{Samudra, Patek}, 0.386	{Samudra, Feri, Azahari}, 0.430	{Samudra, Muklas, Feri, Ghoni}, 0.453
Sarijo, 0.005	{Samudra, Rauf}, 0.384	{Samudra, Muklas, Ghoni}, 0.429	{Samudra, Muklas, Azahari, Sarijo}, 0.429

Taking into account expression (15), we can estimate the interaction between Samudra and Feri, and Samudra with Azahari, as follows:

$$\begin{aligned}\hat{\psi}_{Samudra,Feri}(N,v) &= \hat{MC}_{Feri}(\{Samudra\};N,v)\\ &\quad -\hat{\phi}_{Feri}(N\backslash\{Samudra\},v|_{N\backslash\{Samudra\}})\\ &= -0.008477-0.094708=-0.1032,\\ \hat{\psi}_{Samudra,Azahari}(N,v) &= \hat{MC}_{Azahari}(\{Samudra\};N,v)\\ &\quad -\hat{\phi}_{Azahari}(N\backslash\{Samudra\},v|_{N\backslash\{Samudra\}})\\ &= 0.034-0.03375=0.00025.\end{aligned}$$

Again, the most valuable group of three terrorists is {Samudra, Muklas,Azahari}. In fact, let $C=\{Samudra, Muklas\}$ be the most valuable group of two terrorists, then:

$$\begin{aligned}\hat{\psi}_{Feri,c}(N_C,v_C) &= \hat{MC}_{Feri}(C;N,v)-\hat{\phi}_{Feri}(N\backslash C,v|_{N\backslash C})\\ &= -0.0790-0.1102=-0.1892,\\ \hat{\psi}_{Azahari,v}(N_C,v_C) &= \hat{MC}_{Azahari}(C;N,v)-\hat{\phi}_{Azahari}(N\backslash C,v|_{N\backslash C})\\ &= 0.02402-0.1110=-0.08698.\end{aligned}$$

However, when considering a bigger group of four terrorists, then {Samudra, Muklas, Feri, Azahari} has the highest Shapley group value.

In the analysis of the terrorist network of the 11S, Lindelauf et al.'s starting point was the version of the network in Fig. 4, whose links come from terrorists that lived or learned together (black edges) as well as some temporary links that were only activated just before the attack in order to coordinate the cells. See Krebs (2002) for further information. The authors use the game (N, v^{wconn2}), which uses information about the individuals:

$$v^{wconn2}(S)=\begin{cases}\sum_{i\in S} w_i, & \text{if } S \text{ is connected in } \Gamma \text{ and } |S|>1,\\ 0, & \text{otherwise,}\end{cases}\quad \text{for all } S\subset N, \tag{9}$$

where w_i is the weight assigned to terrorist $i \in N$. The authors also determine the terrorist weights in their analysis (see Table 5 in Lindelauf et al. 2013).

The results obtained by means of Monte carlo simulation (see Castro et al. 2009) are depicted in Table 3, which includes the records for the best groups arranged in decreasing order of importance.

According to the individual rankings for the Al Qaeda's 9/11 network based on the Shapley value, the most valuable terrorists were, in decreasing order of importance: A. Aziz Al-Omari (WTC North cell), H. Al-Ghamdi (WTC South cell), Wd. Al-Shehri (WTC North cell), H. Hanjour (Pentagon cell), M. Al-Shehhi (WTC South cell) and M. Atta (WTC North cell).

With respect of groups of two terrorists, the most valuable group is that composed by the two most important agents, C_2^1 ={A. Aziz Al-Omari, Al-Ghamdi}. However,

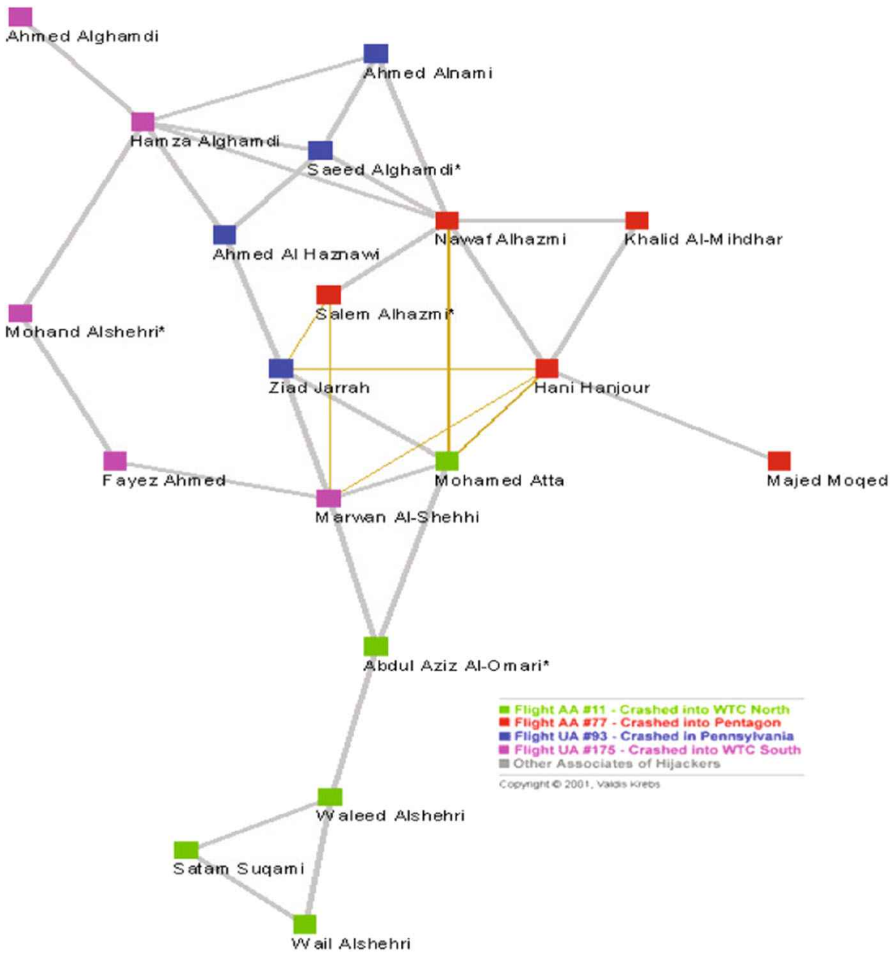

Fig. 4 11S Social Network. Image taken from V.E. Krebs (Copyright ©2002, First Monday)

the second group of size two in importance is C_2^2 ={Aziz Al-Omari, M. Al-Sehhi}, improving the Shapley group value of {Aziz Al-Omari, Wd. Al-Shehri}, which takes the 4th place. In fact, Wd. Al-Shehri is one out of the three hijackers that crashed the plane into WTC South which forms a cycle in the terrorist network that is connected to the rest of terrorist only via A. Aziz Al-Omari, who also belongs to the WTC North cell. Thus, Wd. Al-Shehri presence in a group is not so necessary if A. Aziz Al-Omari is already in it. In that case, the interaction between Aziz Al-Omari and M. Al-Sehhi is negative, and thus they are not complements, whereas Al-Omari and Al-Shehri are. However, Al-Sehhi contributes more to Al-Omari than Al-Shehri does because the power of the most valuable terrorist from an individual point of view strongly overlaps with Al-Shehri's power. In fact, the power of Wd. Al-Shehri without Aziz Al-Omari reduces to 1, which represents an 82.02% of reduction; whereas the power of M. Al-Shehhi undergoes an increment of 17.53%.

Table 3 11S-hijackers network rankings

Individuals	Two agents	Three agents
A. Aziz Al-Omari (WTC-N), 6.096	{Al-Omari, Al-Ghamdi}, 7.405	{Al-Omari, H. Hanjour, M. Al-Shehhi}, 9.236
H. Al-Ghamdi (WTC-S), 5.578	{Al-Omari, M. Al-Shehhi}, 7.392	{Al-Omari, H. Al-Ghamdi, Wd. Al-Shehri}, 9.153
Wd. Al-Shehri (WTC-N), 5.563	{Al-Omari, H. Hanjour}, 7.368	{M. Al-Shehhi, H. Al-Ghamdi, Wd. Al-Shehri}, 9.140
H. Hanjour (Pent), 5.402	{Aziz Al-Omari, Wd. Al-Shehri}, 7.324	{H. Al-Ghamdi, Wd. Al-Shehri, H. Hanjour}, 9.074
M. Al-Shehhi (WTC-S), 2.202	{Al-Omari, M. Atta}, 7.156	{H. Al-Ghamdi, H. Hanjour, N. Al-Hazmi}, 8.986
M. Atta (WTC-North), 1.600	{H. Al-Ghamdi, H. Hanjour}, 7.044	{Al-Omari, H. Al-Ghamdi, H. Hanjour}, 8.963

$$\begin{aligned}&\hat{\psi}_{\textit{Al-Omari,Al-Shehhi}}(N, v)\\&\quad= \hat{MC}_{\textit{Al-Shehhi}}(\{\textit{Al-Omari}\}; N, v) - \hat{\phi}_{\textit{Al-Shehhi}}(N\backslash\{\textit{Al-Omari}\}, v|_{N\backslash\{\textit{Al-Omari}\}})\\&\quad= 1.2128 - 2.5886 = -1.3758,\end{aligned}$$

and

$$\begin{aligned}&\hat{\psi}_{\textit{Al-Omari,Al-Shehri}}(N, v)\\&\quad= \hat{MC}_{\textit{Al-Shehri}}(\{\textit{Al-Omari}\}; N, v) - \hat{\phi}_{\textit{Al-Shehri}}(N\backslash\{\textit{Al-Omari}\}, v|_{N\backslash\{\textit{Al-Omari}\}})\\&\quad= 1.1399 - 1 = 0.1399.\end{aligned}$$

The most valuable group of three terrorists is C_3^1 ={A. Aziz Al-Omari, H. Hanjour, M. Al-Shehhi}= $C_2^2 \cup$ {Hanjour}. In that case, Wd. Al-Shehri, from WTC North cell, and H. Al-Ghamdi, from WTC South cell, are displaced. Now, H. Hanjour, who is known to be the leader of WTC South cell, has displaced H. Al-Ghamdi. The following figures shed some light on what is going on. Recall that the power of Wd. Al-Shehri without Aziz Al-Omari reduces drastically. With respect to the remaining combinations, the one that results in C_3^1 has the greatest interaction index, showing that in both cases Hanjour is less replaceable by each incumbent group than Al-Ghamdi and Al-Shehhi.

$$\begin{aligned}\hat{\psi}_{\textit{Al-Shehri},c_2^1}(N_{C_2^1}, v_{C_2^1}) &= \hat{MC}_{\textit{Al-Shehri}}(C_2^1; N, v) - \hat{\phi}_{Al-Shehri}(N\backslash C_2^1, v|_{N\backslash C_2^1})\\&= 1.7371 - 1 = 0.7371,\\ \hat{\psi}_{Hanjour,c_2^1}(N_{C_2^1}, v_{C_2^1}) &= \hat{MC}_{Hanjour}(C_2^1; N, v) - \hat{\phi}_{Hanjour}(N\backslash C_2^1, v|_{N\backslash C_2^1})\\&= 1.5606 - 5.7129 = -4.1523,\\ \hat{\psi}_{\textit{Al-Shehhi},c_2^1}(N_{C_2^1}, v_{C_2^1}) &= \hat{MC}_{\textit{Al-Shehhi}}(C_2^1; N, v) - \hat{\phi}_{\textit{Al-Shehhi}}(N\backslash C_2^1, v|_{N\backslash C_2^1})\\&= 1.2320 - 6.4177 = -5.1857,\end{aligned}$$

and

$$\begin{aligned}\hat{\psi}_{\textit{Al-Ghamdi},c_2^2}(N_{C_2^2}, v_{C_2^2}) &= \hat{MC}_{\textit{Al-Ghamdi}}(C_2^2; N, v) - \hat{\phi}_{\textit{Al-Ghamdi}}(N\backslash C_2^2, v|_{N\backslash C_2^2})\\&= 1.2403 - 6.3957 = -5.1554,\\ \hat{\psi}_{\textit{Al-Shehri},c_2^2}(N_{C_2^2}, v_{C_2^2}) &= \hat{MC}_{\textit{Al-Shehri}}(C_2^2; N, v) - \hat{\phi}_{\textit{Al-Shehri}}(N\backslash C_2^1, v|_{N\backslash C_2^1})\\&= 1.7414 - 1 = 0.7414,\\ \hat{\psi}_{Hanjour,c_2^2}(N_{C_2^2}, v_{C_2^2}) &= \hat{MC}_{Hanjour}(C_2^2; N, v) - \hat{\phi}_{Hanjour}(N\backslash C_2^1, v|_{N\backslash C_2^1})\\&= 1.8349 - 5.2388 = -3.4039.\end{aligned}$$

When considering a bigger group of four terrorists, then {A. Aziz Al-Omari, H. Hanjour, M. Al-Sehhi, Wd. Al-Shehri} has the highest Shapley group value. The first group with one representative for each cell, being {A. Aziz Al-Omari, H. Hanjour, M. Al-Sehhi, Z. Jarrah}, occupies the 13th place, with a Shapley group value of $\phi^g(D; N, w^{conn2}) = 10.6311$. Note that Z. Jarrah, who is known to be the leader of

Table 4 11S extended network rankings

Individuals	Two agents
N. Al-Hazmi (Pent), 6.132	{N. Al-Hazmi, M. Atta}, 8.424
H. Hanjour (Pent), 6.089	{N. Al-Hazmi, H. Hanjour}, 8.342
M. Atta (WTC-N), 5.926	{ H. Hanjour, M. Atta}, 8.152
H. Al-Ghamdi (WTC-S), 1.844	{N. Al-Hazmi, H. Al-Ghamdi}, 7.201
Wd. Al-Shehri (WTC-N), 1.701	{H. Hanjour, Z. Jarrah}, 7.048
Z. Jarrah (Penn), 1.688	{M. Atta, Z. Jarrah}, 7.013

the Pennsylvania cell is individually in the 8th position. The group of the four cell's leaders $L = \{$M. Atta, H. Hanjour, M. Al-Sehhi, Z. Jarrah$\}$ is in the 48th position with a Shapley group value of $\phi^g(L; N, w^{conn2}) = 9.1313$.

Recall that Lindelauf et al. (2013) carried out the analysis on the terrorist network of the nineteen hijackers which prepared and executed the attack (distributed in four cells). We extend their analysis to a more dense network (see Fig. 5) which included some people which did not take direct part in the attack, but support the terrorists. In this event, the relative positions of the hijackers change: the two poor connected terrorists from the WTC North cell, A. Aziz Al-Omari and Wd. Al-Shehri, are not so relevant in the new network, since they are now better connected through non-hijackers terrorists. According to the rankings for the Al Qaeda's 9/11 hijackers based on the individual Shapley value and the extended network,[8] the most valuable hijackers were, in decreasing order of importance, N. Al-Hazmi (Pentagon), H. Hanjour (Pentagon), M. Atta (WTC North), H. Al-Ghamdi (WTC South), Wd. Al-Shehri (WTC North) and Z. Jarrah (Pennsylvania).

The results obtained by means of Monte carlo simulation (see Castro et al. 2009) are depicted in Table 4, which includes the records for the best groups arranged in decreasing order of importance.

With respect to groups of two terrorists, the most valuable group is {N. Al-Hazmi, M. Atta}, which does not coincide with the two most important terrorists' group. In that case, however, the three and four people most valuable 3-group and 4-group are composed of the three and four, respectively, most important hijackers from an individual point of view. Those groups, C_3 and C_4, have Shapley group values of $\phi^g(C_3; N \cup M, w^{conn2}) = 10.675$ and $\phi^g(C_4; N \cup M, w^{conn2}) = 12.529$. Now, the group L is in the twentieth position with a value of $\phi^g(L; N \cup M, w^{conn2}) = 10.066$.

6 Conclusions

The main motivation of this work has been to apply a (marginalistic) extension of the Shapley value to the problem of evaluation of the prospects of a group of players in a multi-person game. Following the original formulation of Shapley, who apply his

[8] In which we have again considered the w^{conn2} game, with a zero weight for all the terrorists who do not take direct part in the attacks.

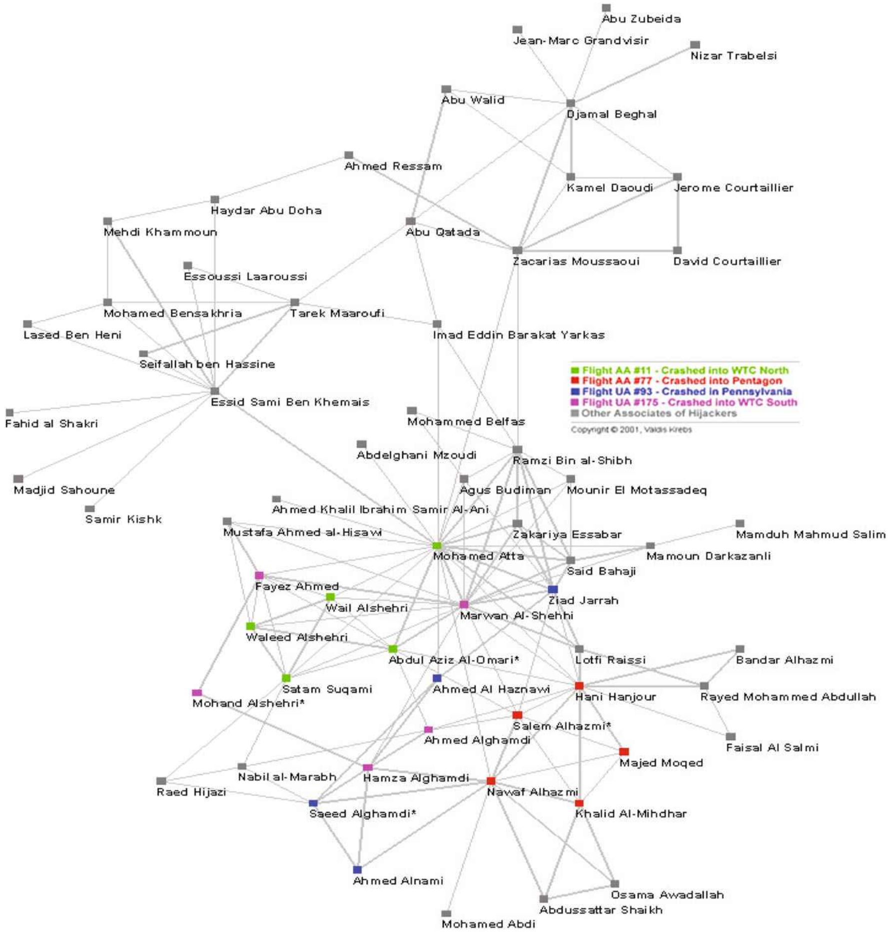

Fig. 5 11S extended SN. Image taken from V.E. Krebs (Copyright ©2002, First Monday)

value to measure the expectations of players in a game; and also keeping in mind the mentioned idea of the external decision maker, we have re-interpreted the generalized Shapley value of Marichal et al. (2007) by means of the merging game defined by Derks and Tijs (2000). In this work, the authors develop a concept of super-player, who in a merging game, acts as a proxy of all the players of the coalition whose value we do want to compute. In order to show that this extension of Shapley (which we call in this context "Shapley group value") is valid and interesting, we have studied the properties of the value, and in particular we have given an axiomatic characterization of it. This axiomatic cannot be directly deduced from the usual characterizations in the individual case, and does not contain a direct formulation of the classical individual efficiency axiom. We also analyze how it behaves as a method to evaluate the effectiveness of a given set of agents to promote a given innovation or behavior in a Social Network, in order to select a *key group* of k agents.

Based on our proposal, and following ideas of Segal, we have elaborated about the ideas of complementarity and substitutability that concerns to profitability of acting as a group. However, it is not easy to find a straight relation between complementarity/substitutability and profitability (or not) because this relation only could be found by means of Segal third derivative, and not using only the second derivative.

Our work culminates by testing the validity of our methods in the identification of influent groups inside a real terrorist network. The flexibility of the proposed approach allows to suppose that our measure will be effective and usual in a variety of contexts and making use of different interpretations of the Shapley group value. Let us think for instance in two of them.

In a global economy context, in which many firms present a complex interlocked shareholding structure, it may be difficult to asses a firm's controllers. However, "a common intuition among scholars and in media sees the global economy as being dominated by a handful of powerful transnational corporation" (Vitali et al. 2011). In that case, following a game theoretical approach, we can make use of the Shapley group value to detect a small group of firms which in fact have a dominating power. The reader is referred to Crama and Leruth (2013) for an interesting review about this approach.

Another relevant application arises in the context of a transportation network's operation, where the identification of sets of stations that should be defended (or maintained), in order to maximally preserve the network's operation, is a relevant question to network protection against natural and human-caused hazards. This has become a typical research topic in engineering and social sciences, as Liu et al. (2009) point out. In that case, following a game theoretical approach, the Shapley group value can serve to security agencies for selecting a group of stations to defend (or maintain). However, it should be remarked that the problem of finding the optimal group, according to some prearranged criteria, is a combinatorial problem that merits a more careful study. We are aware of the need of heuristics in order to apply the Shapley group value to the group selection problem.

Acknowledgements We would like to warmly thank Javier Castro (Universidad Complutense de Madrid) for the simulation program used to obtain the numerical results in Sect. 5.

Appendix

Theorem 1 *The unique group value over the set of all games $\mathcal{G}$ verifying G-null player, G-linearity, G-CBC, and G-SPB is the Shapley group value ϕ^g.*

Proof Let us first prove that the Shapley group value ϕ^g satisfies the previous four properties.

Let us arbitrarily fix the sets C and N, $C \subset N \in \mathcal{N}$, and let $v \in \mathcal{G}_N$.

To check properties $P1$ and $P2$ the reader is referred to Marichal et al. (2007).

With respect to property $P3$, G-CBC, let $C \subset N \in \mathcal{N}$ be any two finite sets, and let v any game in $\mathcal{G}_N$. First let us remark that condition (3) is equivalent to:

$$\begin{aligned}&\xi^g(C \cup i; N, v) - \xi^g(C \cup j; N, v)\\ &\quad = (\xi^g(C \cup i; N\backslash j, v_{-j}) - \xi^g(C; N\backslash j, v_{-j}))\\ &\quad\quad -(\xi^g(C \cup j; N\backslash i, v_{-i}) - \xi^g(C; N\backslash i, v_{-i})). \end{aligned} \tag{10}$$

Also note that, by definition of the merging game and the Shapley value, for any $i, j \in N\backslash C$ the following equalities hold:

$$\begin{aligned}\phi^g(C \cup i; N, v) = &\sum_{\substack{S \subset N\backslash C\\ i,j \notin S}} \Big(\frac{s!(n-c-s-1)!}{(n-c)!}(v(S \cup C \cup i) - v(S))\\ &+ \frac{(s+1)!(n-c-s-2)!}{(n-c)!}(v(S \cup C \cup i \cup j) - v(S \cup j))\Big),\\ \phi^g(C; N\backslash j, v_{-j}) = &\sum_{\substack{S \subset N\backslash C\\ i,j \notin S}} \Big(\frac{s!(n-c-s-1)!}{(n-c)!}(v(S \cup C) - v(S))\\ &+ \frac{(s+1)!(n-c-s-2)!}{(n-c)!}(v(S \cup C \cup i) - v(S \cup i))\Big),\\ \phi^g(C \cup j; N\backslash i, v_{-i}) = &\sum_{\substack{S \subset N\backslash C\\ i,j \notin S}} \frac{s!(n-c-s-2)!}{(n-c-1)!}(v(S \cup C \cup j) - v(S)). \end{aligned}$$

Analogous expressions hold for $\phi^g(C \cup j; N, v)$, $\phi^g(C; N\backslash i, v_{-i})$ and $\phi^g(C \cup i; N\backslash j, v_{-j})$. Now it is enough to check that for every $S \in N$ with $i, j \notin S$ the coefficients of $v(S \cup C \cup i \cup j)$, $v(S \cup C \cup i)$, $v(S \cup C \cup j)$, $v(S \cup C)$, $v(S \cup i)$, $v(S \cup j)$ and $v(S)$ are the same in both sides of the equation in (10), and this is easily deduced from the previous expressions. We leave the details to the reader.

It remains property $P4$, G-SPB. Consider the unanimity game with respect to the grand coalition (N, u_N), and a non-empty group $C \subset N \in \mathcal{N}$. It is straightforward to see that the merging game $(N_C, (u_N)_C)$ is the unanimity game (N_C, u_{N_C}) with respect to the grand coalition N_C, so $\phi^g(C; N, u_N) := \phi_{\mathbf{c}}(N_C, (u_N)_C) = \frac{1}{n-c+1}$, as desired. □

Proof We have proved that the properties hold for the Shapley group value, so we are left with the question of **uniqueness**.

Since $\{(N, u_S)\}_{\substack{S \subset N\\ S \neq \varnothing}}$ forms a basis of $\mathcal{G}_N$ for all $N \in \mathcal{N}$, by G-linearity it is sufficient to consider the games (N, u_S), $\varnothing \neq S \subset N \in \mathcal{N}$. So let us see that $\xi^g(C; N, u_S) = \phi^g(C; N, u_S) := \phi_{\mathbf{c}}(N_C, (u_S)_C)$ for all non-empty subsets $C, S \subset N \in \mathcal{N}$. When $C = \varnothing$, this equality trivially holds by definition of a group value.

The proof will consist in a double induction over the cardinality of the player set N (first induction) and the cardinality of the unanimous coalition $S \subset N$ (second induction).

First, we will prove that $\xi^g(C; N, u_S) = \phi^g(C; N, u_S)$ for all non-empty subsets $C, S \subset N$ whenever the cardinality of $N \in \mathcal{N}$ is $n \leq 2$. For the unanimity game $(\{i\}, u_i)$ with just one player $i \equiv N$, G-SPB property $P4$ implies that $\xi^g(i; N, u_i) = 1 = \phi^g(i; N, u_i)$. Now, let (N, u_S) be a two-person unanimity game with $N = \{i, j\}$. For the unanimity game u_N, $P4$ implies that $\xi^g(\{i, j\}; N, u_N) = \phi^g(\{i, j\}; N, u_N)$. For the unanimity game (N, u_i), G-null player $P1$ implies:

$$\xi^g(j; N, u_i) = \xi^g(\varnothing; N, u_i) := 0 \text{ and } \xi^g(i; N, u_i) = \xi^g(\{i, j\}; N, u_i), \qquad (11)$$

Now, G-CBC property $P3$ implies

$$\xi^g(i; N, u_i) - \xi^g(j; N, u_i) = (\xi^g(i; \{i\}, u_i) - \xi^g(\varnothing; \{i\}, u_i) - (\xi^g(j; \{j\}, u^0) - \xi^g(\varnothing; \{j\}, u^0)), = 1 - 0 - 0 + 0,$$

where $\xi^g(i; N, u_i) = 0$ for the trivial game $(\{i\}, u^0)$ with $u^0(i) = 0$, follows from $P1$. Therefore, taking into account (11), $\xi^g(\cdot; \{i, j\}, u_i) \equiv \phi^g(\cdot; \{i, j\}, u_i)$ holds. The same reasoning applies to (N, u_j).

Let us fix a player set $N \in \mathcal{N}$ with $|N| = r$, and consider the unanimity game (N, u_S) for a fix set $\varnothing \neq S \subset N$. We will prove that $\xi^g(C; N, u_S) = \phi^g(C; N, u_S)$ for all $C \subset N$. Two cases are possible:

(i) If $S = N$, then G-SPB implies $\xi^g(C; N, u_N) = \phi^g(C; N, u_N)$ for any non-empty group C in N.

(ii) Otherwise, if $\varnothing \neq S \subsetneq N$, we proceed by induction on the cardinality of C. Let us first prove the individual case $C = \{i\}$.
There is at least a player $j \in N \backslash S$ which by definition of unanimity game must be null. Let i be a player in S. Again by G-CBC, taking $C = \varnothing$, we obtain $\xi^g(i; N, u_S) = \xi^g(i; N \backslash j, u_S|_{N \backslash j})$, since G-null player implies $\xi^g(j; N, u_S) = \xi^g(\varnothing; N, u_S) = 0$, and taking into account that $u_S|_{N \backslash i} \equiv u^0 \in \mathcal{G}_{N \backslash i}$.
Now, we may assume by the first induction that for every unanimity game $(N', u_S|_{N'})$ with $N' \subsetneq N$ we have $\xi^g(C; N', u_S|_{N'}) = \phi^g(C; N', u_S|_{N'})$ for any group C in N'. Thus, $\xi^g(i; N \backslash j, u_S|_{N \backslash j}) = \phi^g(i; N \backslash j, u_S|_{N \backslash j})$, which in turn is equal to $\frac{1}{s}$ by definition, and then $\xi^g(i; N, u_S) = \frac{1}{s} = \phi_i(N, u_S) = \phi^g(i; N, u_S)$. Note that every $i \notin S$ is a null player in u_S and therefore G-null player implies $\xi^g(i; N, u_S) = 0 = \phi^g(i; N, u_S)$. So we are done with the individual case $C = \{i\}$.
Now, in order to prove $\xi^g(C; N, u_S) = \phi^g(C; N, u_S)$ for all groups C with $c > 1$, we proceed by induction on the cardinality of C (second induction). So we take now $1 < r' \leq r$, and we may assume that $\xi^g(C; N, u_S) = \phi^g(C; N, u_S)$ holds for any $C \subset N$ with $|C| < r'$. We will check that $\xi^g(D; N, u_S) = \phi^g(D; N, u_S)$ for all $D \subset N$ with $|D| = r'$.
Let D be a fixed subset of N of cardinality r'. Since $S \subsetneq N$, again there is a null player j in u_S. So, if $j \in D$, then $D \backslash j$ is a coalition of cardinal $r' - 1$ and, thus, G-null player and the second induction hypothesis imply

$$\xi^g(D; N, u_S) = \xi^g(D \backslash j; N, u_S) = \phi^g(D \backslash j; N, u_S) = \phi^g(D; N, u_S).$$

Otherwise, if D does not contain any null player, then let i be a player in $D \subset S$. By the second induction hypothesis and G-null player property it holds

$$\xi^g((D\backslash i) \cup j; N, u_S) = \xi^g((D\backslash i); N, u_S) = \phi^g((D\backslash i); N, u_S) \\ = \phi^g((D\backslash i) \cup j; N, u_S).$$

Note also that $\xi^g(C; N\backslash i, u_S|_{N\backslash i})) = 0 = \phi^g(C; N\backslash i, u_S|_{N\backslash i}))$ for all $C \subset N$, since $u_S|_{N\backslash i} \equiv u^0 \in \mathcal{G}_{N\backslash i}$. Hence, by G-CBC, taking $C = D\backslash i$, and the first induction,

$$\begin{aligned}
&\xi^g(D; N, u_S) - \phi^g((D\backslash i) \cup j; N, u_S) \\
&\quad = (\phi^g(D; N\backslash j, u_S|_{N\backslash j}) - \phi^g(D\backslash i; N\backslash j, u_S|_{N\backslash j})) \\
&\qquad -(\phi^g((D\backslash i) \cup j; N\backslash i, u_S|_{N\backslash i}) - \phi^g(D\backslash i; N\backslash i, u_S|_{N\backslash i})) \\
&\quad = \phi^g(D; N, u_S) - \phi^g((D\backslash i) \cup j; N, u_S).
\end{aligned}$$

So we have proved the uniqueness for the unanimity games (N, u_S) for all $S \subset N \in \mathcal{N}$ and we are done. □

Aside from the previous considerations regarding the alternative axioms of Marichal et al. (2007), it must be remarked that it is not a trivial extension of a characterization of the Shapley value, since we do not impose any condition about the value of the individual agents out of the group C we are evaluating. In particular, we have been forced to use in the same characterization group linearity and group coalitional balanced contributions properties. In the following we will check that all the axioms above are necessary to guarantee the uniqueness of the Shapley group value ϕ^g.

G-**null player**. Let $\alpha \in (0, 1)$. Define a value ξ^g in the following manner. If (N, v) is a null game, $\xi^g_C(N, v)$=0 for every $C \in N$. Given a non-null unanimity game u_S, with $S \subsetneq N$, define $\xi^g_C(u_S)$ as

$$\xi^g_C(N, u_S) = \begin{cases} \displaystyle\sum_{k=0}^{c-1} \alpha^{n-k}, & \text{if } C \cap S = \varnothing, \\ \displaystyle\frac{1}{s - |S \cap C| + 1} + \sum_{k=0}^{n-s-1} \alpha^{n-k}, & \text{if } C \cap S \neq \varnothing, \end{cases} \tag{12}$$

$\xi^g_C(N, u_N) = \frac{1}{n-c+1}$, for all $C \subseteq N$, and then extend the value by additivity. It is clear that $\xi^g_i(N, u_S) = \alpha^n > 0$ for all $i \notin S$. So, G-null player does not hold. Let us check that ξ^g verifies G-CBC over the class of unanimity games. If (N, u_N), then G-CBC condition (10) trivially holds. Let (N, u_S) be a unanimity game with $S \subsetneq N$. If $|S| = n - 1$, then two cases are possible:

(a) If $i, j \in S$, then $\xi^g_{C\cup i}(N, u_S) = \frac{1}{s-|S\cap C|} + \sum_{k=0}^{n-s-1} \alpha^{n-k} = \xi^g_{C\cup j}(N, u_S)$ and (10) holds, since $u_S|_{N\backslash i}$ and $u_S|_{N\backslash j}$ are null games.

(b) If $i \in S$ and $j \notin S$, then $S = N\backslash j$, $u_S|_{N\backslash i} \equiv 0$ and $u_S|_{N\backslash j}$ is a unanimity game w.r.t. the grand coalition $N\backslash j$. Thus (10) holds:

$$\xi^g_{C\cup i}(N\backslash j, u_{N\backslash j}|_{N\backslash j}) - \xi^g_C(N\backslash j, u_{N\backslash j}|_{N\backslash j})$$
$$= \frac{1}{(n-1)-(c+1)+1} - \frac{1}{(n-1)-c+1} = \xi^g_{C\cup i}(N, u_{N\backslash j}) - \xi^g_{C\cup j}(N u_{N\backslash j}).$$

If $|S| < n-1$, then three cases are possible:

(a) If $i, j \in S$, then $\xi^g_{C\cup i}(N, u_S) = \frac{1}{s-|S\cap C|} + \sum_{k=0}^{n-s-1} \alpha^{n-k} = \xi^g_{C\cup j}(N, u_S)$ and (10) holds.

(b) If $i \in S$ and $j \notin S$, then

$$\xi^g_{C\cup i}(N\backslash j, u_S|_{N\backslash j}) = \frac{1}{s-|S\cap C|} + \sum_{k=1}^{n-s-1} \alpha^{n-k}$$

and

$$\xi^g_C(N\backslash j, u_S|_{N\backslash j}) = \begin{cases} \sum_{k=1}^{c} \alpha^{n-k}, & \text{if } C \subseteq N\backslash S, \\ \frac{1}{s-|S\cap C|+1} + \sum_{k=1}^{n-s-1} \alpha^{n-k}, & \text{otherwise.} \end{cases}$$

Thus, if $C \subseteq N\backslash S$:

$$\xi^g_{C\cup i}(N\backslash j, u_S|_{N\backslash j}) - \xi^g_C(N\backslash j, u_S|_{N\backslash j})$$
$$= \frac{1}{s} + \sum_{k=1}^{n-s-1} \alpha^{n-k} - \sum_{k=1}^{c} \alpha^{n-k} = \xi^g_{C\cup i}(N, u_S) - \xi^g_{C\cup j}(N, u_S)$$

If $C \cap S \neq \varnothing$, then

$$\xi^g_{C\cup i}(N\backslash j, u_S|_{N\backslash j}) - \xi^g_C(N\backslash j, u_S|_{N\backslash j})$$
$$= \frac{1}{s-|S\cap C|} + \sum_{k=1}^{n-s-1} \alpha^{n-k} - \left(\frac{1}{s-|S\cap C|+1} + \sum_{k=1}^{n-s-1} \alpha^{n-k}\right.$$
$$\left.= \xi^g_{C\cup i}(N, u_S) - \xi^g_{C\cup j}(N, u_S)\right)$$

c) If $i, j \in N\backslash S$, then $C \cup i \subseteq N\backslash S$ if, and only if, $C \cup j \subseteq N\backslash S$ and, therefore condition (10) can be easily checked.

Now, since G-CBC condition is additive ξ^g satisfies it over $\bigcup_{n\geq 1} \mathcal{G}_n$
G-**linearity**. Let ξ^g be another group value over (N, v) which is defined in the following way:

(A$_1$) If there is at least a null player in N, or (N, v) is the unanimity game with respect to the grand coalition N, then $\xi^g_C(N, v) = \phi^g_C(N, v)$ for every group C in N.

(A$_2$) Otherwise, $\xi^g_C(N, v) = \phi^g_C(N, v) + k$, being $k \neq 0$ a fixed constant.

It is easily checked that G-null-player and G-SPB hold for ξ^g. We will check that the property of coalitional balanced contributions G-BMC also holds for this value. Note that all differences in (10) match $\xi^g_D(L, w)$ with $\xi^g_{D'}(L, w)$, for some coalition $L \in \{N, N\backslash i, N\backslash j\}$ and some game $w \in \{v, v_{-i}, v_{-j}\}$.

Taking into account that $\phi^g = \xi^g$ in case (N, v) is some of the games in the first case (A_1), and in the second one the $k's$ cancel, it holds:

$$\xi^g_{C\cup i}(N, v) - \xi^g_{C\cup j}(N, v) = \phi^g_{C\cup i}(N, v) - \phi^g_{C\cup j}(N, v),$$
$$\xi^g_{C\cup i}(N\backslash j, v_{-j}) - \xi^g_C(N\backslash j, v_{-j}) = \phi^g_{C\cup i}(N\backslash j, v_{-j}) - \phi^g_C(N\backslash j, v_{-j}),$$
$$\xi^g_{C\cup j}(N\backslash i, v_{-i}) - \xi^g_C(N\backslash i, v_{-i}) = \phi^g_{C\cup j}(N\backslash i, v_{-i}) - \phi^g_C(N\backslash i, v_{-i}),$$

G-CBC property holds for ϕ^g, and so the property does so for ξ^g, and we are done.

Note that we can modify ξ^g defining $\xi^g_i(N, v) = \Phi_i(N, v)$, for all $i \in N$, and $\xi^g_N(N, v) = v(N)$, for all $N \subseteq \mathbb{N}$, and the same result holds.

G-**CBC**. Define a value ξ^g in the following manner. If (N, v) is a null game, $\xi^g_C(N, v)$=0 for every $C \in N$. Given a non-null unanimity game u_S with $S \subsetneq N$, define $\xi^g_C(u_S)$ as

$$\xi^g_C(u_S) = \begin{cases} 0 & \text{if } C \cap S = \varnothing, \\ \xi^g_C(u_S) = 1, & \text{if } S \subseteq C, \\ |S \cap C| \times |S\backslash C| & \text{if } C \cap S \neq \varnothing \text{ and } C \cap S \neq S, \end{cases} \tag{13}$$

$\xi^g_C(N, u_N) = \frac{1}{n-c+1}$, for all $C \subseteq N$, and then extend the value by additivity.

Observe that all axioms but G-CBC hold. The unique one that needs a bit of discussion is the G-null player axiom, which holds when considering the base of unanimity games because in u_S the null players are precisely the players outside S, and therefore:

$$\begin{aligned} &\xi^g_{C\cup i}(N, u_S) = 0 = \xi^g_C(N, u_S), \text{ if } S \cap C = \varnothing, \\ &\xi^g_{C\cup i}(N, u_S) = 1 = \xi^g_C(N, u_S), \text{ if } S \subseteq C, \text{ and} \\ &\xi^g_{C\cup i}(N, u_S) = |S \cap (C \cup i)| \times |S\backslash(C \cup i)| = |S \cap C| \times |S\backslash C| = \xi^g_C(N, u_S), \text{ otherwise,} \end{aligned}$$

for all player $i \notin S$. Then, taking into account that the Harsanyi dividend $c_S(N, v)$ of any coalition S containing null players in the game (N, v) equals zero, G-null player property holds for any n-person game $(N, v) \in \mathcal{G}_n$, for all $N \subseteq \mathbb{N}$.

Let us check by means of a concrete example that the G-CBC axiom fails in this case. Consider (N, u_S) with $|N| = 3$, and $S = \{1, 2\}$. In the notation of the axiom, take $C = \{1\}$, $i = 2$, and $j = 3$. Then:

$$\begin{aligned} \xi^g_{\{1,2\}}(N, u_S) = 1,\ \xi^g_{\{1,3\}}(N, u_S) = 1 \times 1 = 1,\ &\xi^g_{\{1,2\}}(N\backslash 3, u_S|_{N\backslash 3}) \\ &= 1,\ \xi^g_1(N\backslash 3, u_S|_{N\backslash 3}) = 1/2, \end{aligned}$$

and $\xi^g_{\{1,3\}}(N\backslash 2, u_S|_{N\backslash 2}) = 0 = \xi^g_1(N\backslash 2, u_S|_{N\backslash 2})$, because the game $(N\backslash 2, u_S|_{N\backslash 2})$ is null. It is clear now that the two sides of the equalities that define the axiom do not coincide in this case.

Note that we can modify ξ^g defining $\xi^g_i(N, u_S) = \Phi_i(N, v)$, for all $i \in N$, and $\varnothing \neq S \subsetneq N$, for all $N \subseteq \mathbb{N}$, and the same result holds. Probably it is easy to find more examples by defining the value over $C \cap S$ (when $C \cap S \neq \varnothing$) as another appropriate function of $(|C \cap S|, |S\backslash C|)$.

G-**symmetry over pure bargaining games**. Define a value ξ^g in the following manner. If (N, v) is a null game, $\xi^g_C(N, v)$=0 for every $C \in N$. Given a non-null unanimity game u_S, define $\xi^g(N, u_S)$ as $\xi^g_C(N, u_S) = \frac{|C \cap S|}{|S|}$, for all group $C \subseteq N$, and then extend the value by additivity.

Observe that all axioms but *G*-SPB hold. The *G*-null player axiom trivially holds when considering the base of unanimity games. Then, since $c_S(N, v) = 0$ for all S containing null players in the game (N, v), *G*-null player property holds in general. Moreover, *G*-additivity follows from the definition.

Let us check that the *G*-CBC axiom holds. Let (N, v) be any n-person game with $n \geq 2$. Let C be any group in N of cardinality $c \leq n - 2$, and let $i, j \in N\backslash C$. Then:

- If $i, j \in N\backslash S$, then (10) holds since all the involved differences are zero because i and j are null players in the three games.
- If $i, j \in S$, then $|(C \cup i) \cap S| = |C \cap S| + 1 = |(C \cup j) \cap S|$. Thus, $\xi^g_{C\cup i}(N, u_S) - \xi^g_{C\cup j}(N, u_S) = 0$, and (10) holds because the games $(N\backslash i, u_S|_{N\backslash i})$ and $(N\backslash j, u_S|_{N\backslash j})$ are null.
- If $i \in S$ and $j \notin S$, then $\xi^g_{C\cup i}(N, u_S) - \xi^g_{C\cup j}(N, u_S) = \frac{1}{s} = \xi^g_{C\cup i}(N\backslash j, u_S|_{N\backslash j}) - \xi^g_C(N\backslash j, u_S|_{N\backslash j})$, and (10) holds because $u_S|_{N\backslash i} \equiv 0$.

Clearly, *G*-SPB fails, so we are done.

Proposition 1 *Let $N \in \mathcal{N}$ be any finite set of players, and v be any game in $\mathcal{G}_N$. Then, the Shapley group value ϕ^g verifies the following properties:*

(i) Group Rationality*: $\phi^g(C; N, v) \geq v(C)$ for every $C \subset N \in \mathcal{N}$ if the game $v \in \mathcal{G}_N$ is superadditive, and*
(ii) Monotonicity*: $\phi^g(C; N, v) \leq \phi^g(D; N, v)$ for every pair of coalitions $C \subset D \subset N \in \mathcal{N}$ if the game $v \in \mathcal{G}_N$ is monotonic.*

Proof Group rationality follows from the individual rationality of the Shapley value. Note that every merging game (N_C, v_C), $C \subset N$, is superadditive if it is (N, v), for all $C \subset N \in \mathcal{N}$ and $v \in \mathcal{G}_N$.

Monotonicity follows from being

$$\phi^g(C \cup i; N, v) - \phi^g(C; N, v) = \sum_{\substack{S \subset N\backslash C \\ i \notin S}} \frac{s!(n-c-s)!}{(n-c+1)!}\Big(v(S \cup i \cup C) - v(S \cup C)\Big) + \frac{(s+1)!(n-c-s-1)!}{(n-c+1)!}\Big(v(S \cup i) - v(S)\Big) \geq 0, \quad (14)$$

for all coalitions $C \subset N \in \mathcal{N}$, and all players $i \notin C$, whenever the game v is monotonic. □

Theorem 2 *Let $N \in \mathcal{N}$ be any finite set of players, and v be any game in $\mathcal{G}_N$. Let $C \subset N$ be any group in N, and let $i \notin C$. Then, the* marginal contribution *of player $i \in N \backslash C$ to the Shapley group value of C equals:*

$$MC_i^g(C; N, v) := \phi^g(C \cup i; N, v) - \phi^g(C; N, v) = \phi_i(N \backslash C, v|_{N \backslash C}) + \psi_{\mathbf{c}i}(N_C, v_C). \tag{15}$$

Proof Let us arbitrarily fix the sets C and N and the player i, $C \subsetneq N \in \mathcal{N}$, $i \in N \backslash C$, and let $v \in \mathcal{G}_N$. Then, adding and subtracting the amount

$$\sum_{\substack{S \subset N \backslash C \\ i \notin S}} \frac{s!(n-c-s-1)!}{(n-c)!} \Big(v(S \cup i) - v(S) \Big)$$

to the above expression (14) of the marginal contribution $MC_i^g(C; N, v)$ follows that:

$$\begin{aligned} MC_i^g(C; N, v) = & \sum_{\substack{S \subset N \backslash C \\ i \notin S}} \frac{s!(n-c-s-1)!}{(n-c)!} \Big(v(S \cup i) - v(S) \Big) \\ & + \sum_{\substack{S \subset N \backslash C \\ i \notin S}} \frac{s!(n-c-s)!}{(n-c+1)!} \Big(v(S \cup i \cup C) - v(S \cup C) \\ & - v(S \cup i) + v(S) \Big). \end{aligned}$$

The first term is precisely the Shapley value of player i in the restricted game $(N \backslash C, v|_{N \backslash C})$, where players in C do not play a role. The second term can be expressed by means of the second-order difference operators for the pair of players $i, \mathbf{c} \in N_C$, as follows:

$$\begin{aligned} & \sum_{\substack{S \subset N \backslash C \\ i \notin S}} \frac{s!(n-c-s)!}{(n-c+1)!} \Big(v(S \cup i \cup C) - v(S \cup C) - v(S \cup i) + v(S) \Big) \\ & = \sum_{\substack{S \subset N \backslash C \\ i \notin S}} \frac{s!(n-c-s)!}{(n-c+1)!} \Delta_{ic}^2(S; N_C, v_C) =: \psi_{\mathbf{c}i}(N_C, v_C). \end{aligned}$$

□

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