The physics of a push-me pull-you boat

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The basic laws of mass, energy, and momentum conservation are applied to a novel wind-driven water craft. The resulting analysis is an instructive application of these laws, suitable as an undergraduate physics exercise. We find a critical condition which must be met before the boat will accelerate against the wind, and we also obtain expressions for calculating the final speed.

I. INTRODUCTION

A novel wind-driven water craft was recently reported in the literature.[1] In this boat a windmill-type propeller is connected via a straight shaft to an underwater propeller, as shown in Fig. 1. Kinetic energy extracted from the wind by the air propeller is used by the water propeller to move the boat directly *against* the wind. We have demonstrated such a boat at recent public displays in our junior level physics laboratory. A typical nonphysicist is suitably impressed by the novelty of the boat, whereas the initial response of people with some physics background is often one of disbelief. Being familiar with the laws of energy and momentum conservation, they feel that somehow these laws are being violated. Indeed, a detailed analysis of the device is an instructive application of these laws, suitable as an undergraduate physics exercise. (The topic also aroused considerable interest when given to a graduate physics problems class.) We present such an analysis here. It is found that there is a critical condition on the relative sizes of the two propellers which must be met before the boat will move against the wind, even for *ideal* propellers. It is also found that the maximum speed attainable under ideal conditions is twice the wind speed. The performance of an actual boat is expected to be considerably less than this.

II. THEORY

A. Energy from the wind (efficiency of an ideal windmill) (Ref. 2)

Figure 2 shows a propeller of swept out area A_1 immersed in a fluid flow field (density ρ_1) whose undisturbed speed and pressure are W and P_∞, respectively. The air speed at the propeller is $W(1-a)$ where $0 < a < 1$, and the downstream speed is W_∞. If we assume incompressible flow along the streamlines, shown in Fig. 2, then mass conservation requires that

$$AW = A_1 W(1-a) = A_\infty W_\infty, \qquad (1)$$

where A and A_∞ are the cross-sectional areas of the streamtube far upstream and far downstream, respectively. The factor $1-a$ is simply a convenient way of expressing the decrease in wind speed at the propeller. Now apply Bernoulli's theorem, in the form $P + (1/2)\rho v^2 = \text{const}$, to the regions upstream and downstream of the propeller. Upstream of the propeller we have

$$P_\infty + (1/2)\rho_1 W^2 = P_u + (1/2)\rho[W(1-a)]^2, \qquad (2)$$

where P_u is the pressure immediately upstream of the propeller. On the downstream side

$$P_\infty + (1/2)\rho_1 W_\infty^2 = P_d + (1/2)\rho_i[W(1-a)]^2, \qquad (3)$$

where P_d is the pressure immediately downstream of the propeller.

Combining Eqs. (2) and (3) one finds a pressure discontinuity at the propeller, given by $P_u - P_d \equiv \Delta P = (1/2) \times \rho_1(W^2 - W_\infty^2)$. The force exerted on the propeller by the stream is therefore given by

$$F_W \equiv A_1 \Delta P = (1/2)\rho_1 A_1 (W^2 - W_\infty^2). \qquad (4)$$

From momentum conservation, the force on the propeller can also be written

$$F_W = \rho_1 A W^2 - \rho_1 A_\infty W_\infty^2, \qquad (5)$$

where the two terms represent the rates at which momentum enters and leaves the streamtube, respectively. Combining Eqs. (1), (4), and (5) one obtains $W_\infty = W(1-2a)$, which gives

$$F_W = 2\rho_1 A_1 W^2 a(1-a) \qquad (6)$$

when substituted in Eq. (4). The power extracted from the wind is given by

$$P_W = F_W W(1-a) = 2\rho_1 A_1 a(1-a)^2 W^3. \qquad (7)$$

Maximizing the power P_W with respect to the parameter a, gives $a_{\max} = 1/3$. Using this optimum value of a, one finds that the efficiency of the propeller, as defined by $P_W/(1/2)\rho_1 A_1 W^3$, is equal to 16/27. This is the maximum efficiency of an ideal windmill. Real windmills of modern design[2,3] are capable of reaching 60%–70% of this value.

B. Net forward thrust

Assume that the boat is held fixed with respect to the undisturbed water and focus attention on the water propeller A_2, Fig. 1. By using arguments identical to those used above in obtaining Eqs. (1)–(6), one finds that the thrust delivered by the propeller A_2 is given by

$$F_2 = 2\rho_2 A_2 V^2. \qquad (8)$$

In this case, it is helpful to note that the speed of the water in the streamtube far upstream of A_2 is zero while that far downstream is $V_\infty = 2V$, where V is the water speed at the propeller.

The rate of increase of the kinetic energy of the water is given by $F_2 V$ and if this is equated to P_W, as given by Eq. (7), one finds that

$$F_2 = 0.56\, \rho_2 A_2 (\rho_1 A_1/\rho_2 A_2)^{2/3}\, W^2, \qquad (9)$$

where $a = 1/3$ was used. In obtaining Eq. (9), the propeller A_2 was assumed to be capable of converting the shaft energy

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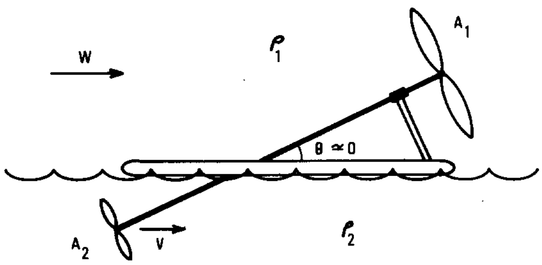

Fig. 1. Sketch of a boat which supports a shaft with a propeller on each end. One propeller, of swept-out area A_1, is located in the air and the other propeller, of area A_2, is located below the water surface. Both propellers are assumed to be far enough from the water surface so that its effect can be ignored. The speed of the air far upstream of the boat is W. The speed of the water immediately downstream of A_2 is V, where V is due to the effects of A_2. ρ_1 and ρ_2 are the densities of air and water, respectively.

to kinetic energy of the water with 100% efficiency. This is *not* inconsistent with the 59% efficiency of an ideal windmill, as it refers to a different problem.

The net forward thrust $F_2 - F_W$ is given by subtracting Eq. (6) from Eq. (9) and rearranging the resulting expression, that is

$$F_{\text{net}} = \rho_1 A_1 W^2[0.56(\rho_2 A_2/\rho_1 A_1)^{1/3} - 4/9]. \quad (10)$$

This expression for F_{net} assumes that both propellers operate at their maximum possible efficiencies and that there are no energy losses in the coupling shaft. It also neglects wind-drag forces on the hull and superstructure of the boat. Even for these ideal conditions we see from Eq. (10) that the net force will be positive only if $\rho_2 A_2/\rho_1 A_1 \geq 1/2$.

The essential qualitative explanation of why the boat moves forward can be seen from Eq. (10). It clearly depends on the way in which the kinetic energy extracted from the air is added to the water. If the streamtube is narrow ($\rho_2 A_2 \ll \rho_1 A_1$) the momentum in the water jet will be less than that extracted form the air and the boat would move backwards. On the other hand, if $\rho_2 A_2 \gg \rho_1 A_1$ the momentum in the water jet is larger than that extracted from the air and a net forward force results. This is analogous to the situation of two masses having the same kinetic energy, whereas the heavier mass has the largest momentum.

C. Ultimate speed

We now consider the situation where the boat is allowed to move forward, with the aim of calculating the ultimate speed u. It is convenient to use a reference frame attached to the boat. The power extracted from the wind by the propeller A_1 will then be given by

$$P'_W = (8/27)\rho_1 A_1 (W+u)^3, \quad (11)$$

where $a = 1/3$ was used in Eq. (7) and the effective wind felt by the boat is $W+u$ rather than W. Similarly, the force on the propeller A_1 is given by

$$F'_W = (4/9)\rho_1 A_1 (W+u)^2. \quad (12)$$

The water propeller A_2 now sees an effective upstream speed equal to u, a relative water speed at the propeller of $u+V$ and a downstream speed equal to $u + V_\infty$. Applying arguments similar to those in Sec. II A, one still finds that $V_\infty = 2V$. By analogy with Eq. (4), the thrust produced by A_2 can then be written

$$F'_2 = (1/2)\rho_2 A_2[(V_\infty + u)^2 - u^2] = 2\rho_2 A_2 (u+V)V. \quad (13)$$

The force F'_2 adds kinetic energy to the water at the rate $F'_2(u+V)$ and if we again assume that the propeller A_2 is 100% efficient then we have, from Eqs. (11) and (13)

$$F'_2(u+V) = 2\rho_2 A_2 (u+V)^2 V = (8/27)\rho_1 A_1 (W+u)^3. \quad (14)$$

At the ultimate speed u all forces on the boat must balance, i.e.,

$$F'_2 - F'_W - F_D = 0, \quad (15)$$

where

$$F_D = (1/2)\rho_2 A_D C_D u^2 \quad (16)$$

is the drag force[4] of the water against the hull. A_D is the effective frontal area of the hull and C_D is the drag coefficient. Substituting Eqs. (12), (14), and (16) into Eq. (15) then gives

$$\frac{(8/27)\rho_1 A_1 (W+u)^3}{(V+u)} - (4/9)\rho_1 A_1 (W+u)^2 - (1/2)\rho_2 A_D C_D u^2 = 0. \quad (17)$$

Equations (14) and (17) represent two equations for the unknowns u and V, which must be solved simultaneously. The general solution is very messy algebraically and it is perhaps more instructive to look at some particular cases.

First, consider the case where the drag coefficient C_D is zero, as this will yield the maximum possible value of u. With this assumption, Eqs. (14) and (17) yield

$$u = \left(\frac{2\rho_2 A_2/\rho_1 A_1 - 1}{\rho_2 A_2/\rho_1 A_1 + 1}\right) W \quad (18)$$

and

$$V = u/[2(\rho_2 A_2/\rho_1 A_1) - 1]. \quad (19)$$

In the limit $\rho_2 A_2/\rho_1 A_1 \gg 1$, $u = 2W$ and $V \ll u$. That is, the maximum possible upwind speed of the device is equal to twice the wind speed. On the other hand, the maximum downwind speed under the same conditions would be equal to the wind speed, as one can easily show from Eqs. (14) and (17) with appropriate sign changes.

The performance of an actual device will be reduced below the maximum value, $u = 2W$, by (a) nonzero drag coefficients, (b) smaller values of the parameter $\rho_2 A_2/\rho_1 A_1$, and (c) propeller inefficiency. For example, if one chooses $A_1 = \pi$ m^2, $A_2 = 0.2$ m^2, $A_D = 0.1$ m^2, $C_D = 0.2$, $\rho_2 = 1000$ kg/m^3, and $\rho_1 = 1.20$ kg/m^3, then one finds from Eqs. (14) and (17), that $u = 0.56W$ and $V = 0.034W$. If propeller

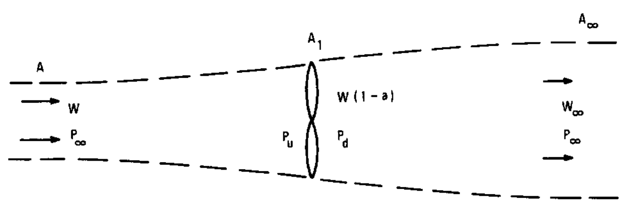

Fig. 2. Sketch of a propeller of swept-out area A_1, immersed in a flowing fluid. The dashed lines depict the streamtube of the fluid which is actually intercepted by the propeller disk. The labeled quantities are defined in the text as needed.

inefficiency is taken into account the value of u will be further reduced, by 50% or more.

D. Energy considerations

In the above derivation of Eq. (14) it was assumed that all of the kinetic energy extracted from the wind is added to the water by the propeller A_2. But, what about the work done by the drag force F_D? Have we violated energy conservation? To answer this question it is best to work in a reference frame at rest with respect to the undisturbed water. In this frame the force F_D moves at speed u and therefore dissipates energy in the water at a rate $F_D u$. If we are to conserve energy then the power extracted from the wind must supply this dissipation as well as the kinetic energy increase in the water passing through the propeller A_2.

First we note that, in the rest frame, the kinetic energy extracted from the wind by A_1 and that added to the water by A_2 will not be the same as in the boat frame. Recall that when a mass m has a speed change $v_f - v_i \equiv \Delta v$ in the rest frame, then the change in kinetic energy is

$$\Delta \mathrm{KE} = (1/2) m (v_f^2 - v_i^2). \quad (19')$$

The corresponding kinetic energy change in a reference frame moving with speed u is given by

$$\Delta \mathrm{KE}' = \Delta \mathrm{KE} + m \Delta v u. \quad (20)$$

Applying this relation to the propellers, in the rest frame, we find that the power extracted from the wind is

$$P_W = P'_W - (4/9) \rho_1 A_1 (u + W)^2 u = P'_W - F_W u, \quad (21)$$

where P'_W is the power from the wind in the moving frame, as given by Eq. (11). F_W is the force of the wind against propeller A_1 and is the same in both frames. Similarly, the power added to the water by A_2 is given by

$$P_2 = P'_2 - 2\rho_2 A_2 (u + V) V u = P'_2 - F_2 u, \quad (22)$$

where P'_2 is the power added to the water by A_2 in the moving frame and F_2 is the force produced by A_2. The energy balance in the rest frame therefore takes the form

$$P_W - P_2 = (P'_W - P'_2) + (F_2 - F_W) u. \quad (23)$$

Recalling that $P'_W = P'_2$ in the moving frame and using Eq. (15), Eq. (23) becomes

$$P_W - P_2 = F_D u. \quad (24)$$

Therefore a satisfactory energy balance does exist in the rest frame.

III. CONCLUDING REMARKS

A successful demonstration boat used a 36-cm-diam model airplane propeller for A_1 and a 10-cm one for A_2. These were connected by a 65 cm × 3.2 mm stainless-steel shaft mounted (shaft inclined at 22° to the horizontal) on a catamaran-type hull made from two thin-walled aluminum tubes (41 × 2.5 cm) separated by 22 cm. Performance can be optimized by trying propellers of various pitch. This unit produced a net forward thrust of nearly 1 N in a windspeed of 5 m/s provided by a household fan.

We note that a vertical-axis windmill[3] could easily be utilized with the above type of boat. The omnidirectional nature of such windmills should give them a practical advantage in this application. We also note that a wheeled vehicle could also be driven against the wind by coupling the propeller shaft to the wheels. The maximum attainable speed, under ideal conditions, is expected to be twice the wind speed in this case also.

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