Perot cavity of $L \sim l \sim 5$ cm and $R \sim 0.6$, for example, the minimum incident power required for the occurrence of a turbulent state is estimated to be 50 MW/cm^2, by using the values $\gamma^{-1} \sim 2$ psec, $n_2 \sim 10^{-11}$ esu and $k \sim 10^5$ cm^{-1}. This value of the incident power can be drastically lowered if a medium with large n_2 is used; 1 kW/cm^2 is sufficient for Rb vapor ($\gamma^{-1} \sim 200$ psec, $n_2 \sim 10^{-6}$ esu) in the same cavity.

An equivalent system whose dynamics obeys Eq. (8) can be constructed by modifying partly the hybrid bistable optical device studied by Garmire et al.[3] One has only to insert a delay line with delay time t_R between the photoconductor which detects the output light from a Pockels cell modulator and the feedback circuit. If the rise time of the detector is short enough, the equation which governs the temporal behavior of the output voltage is identical with Eq. (8), where γ^{-1} should be regarded as the relaxation time of the feedback circuit. In such a system the transition from self-pulsing to turbulence will easily be observed.

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Geometry from a Time Series

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It is shown how the existence of low-dimensional chaotic dynamical systems describing turbulent fluid flow might be determined experimentally. Techniques are outlined for reconstructing phase-space pictures from the observation of a single coordinate of any dissipative dynamical system, and for determining the dimensionality of the system's attractor. These techniques are applied to a well-known simple three-dimensional chaotic dynamical system.

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Lorenz originally demonstrated that very simple low-dimensional systems could display "chaotic" or "turbulent" behavior.[1] Attractors which display such behavior were termed "strange attractors" by Ruelle and Takens,[2] who then went on to conjecture that these strange attractors are the cause of turbulent behavior in fluid flow. The experiments of Gollub and Swinney have strengthened the conjecture,[3] but the question still remains: How can we discern the nature of the strange attractor underlying turbulence from observing the actual fluid flow?

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Data obtained by experimentalists examining turbulent fluid flow often take the form of a "time series," which is to say, a series of values sampled at regular intervals. We address here the problem of using such a time series to reconstruct a finite-dimensional phase-space picture of the sampled system's time evolution. From this picture we can then obtain the asymptotic properties of the system, such as the positive Liapunov characteristic exponents, which are a measure of how chaotic the system is,[4-6] and topological characteristics such as the attractor's topological dimension. We illustrate these reconstruction methods by applying them to a time series obtained from sampling one coordinate of a three-dimensional chaotic dynamical system first studied by Rossler,[7] and then comparing the resulting values of the Liapunov exponents to those obtained by a different method.

The dynamical system of interest is a set of three ordinary differential equations:

$$\dot{x} = -(y + z),$$
$$\dot{y} = x + 0.2y, \qquad (1)$$
$$\dot{z} = 0.4 + xz - 5.7z.$$

These equations have a chaotic attractor which is illustrated in Fig. 1, which was obtained from an analog computer simulation.

The heuristic idea behind the reconstruction method is that to specify the state of a three-dimensional system at any given time, the measurement of *any* three independent quantities should be sufficient, where "independent" is not yet formally defined, but will become operationally defined. We conjecture that any such sets of three independent quantities which uniquely and smoothly label the states of the attractor are diffeomorphically equivalent. The three quantities typically used are the values of each state-space coordinate, $x(t)$, $y(t)$, and $z(t)$. We have found that beginning with a time series obtained by sampling a single coordinate of Eq. (1), one can obtain a variety of three independent quantities which appear to yield a faithful phase-space representation of the dynamics in the original x, y, z space. One possible set of three such quantities is the value of the coordinate with its values at two previous times,[8] e.g., $x(t)$, $x(t - \tau)$, and $x(t - 2\tau)$. Another set obtained by making the time delays small, and taking differences is $x(t)$, $\dot{x}(t)$, and $\ddot{x}(t)$. Figure 2 shows a reconstruction of the $(x, \dot{x})$ picture from the time series taken from sampling the x coordinate of Eq. (1). Comparison of Figs. 1 and 2 certainly indicates that topological characteristics and geometrical form of the attractor remain intact when viewed in the $(x, \dot{x})$ coordinates. For an experimentalist observing some chaotic phenomenon, such as turbulent fluid flow, the construction of phase-space coordinates might not be as simple as the case illustrated above. In many cases the experimentalist has no *a priori* knowledge of how many dimensions a dynamical description would require, nor the quantities appropriate to the construction of such a description. So far there is no universally applicable method of phase-space construction, though the nature of the phenomenon might suggest possible alternatives. In a study of fluid turbulence, for example, the experimentalist might try using the velocity of the fluid in different directions, at different points in space, and at different times.

After having obtained a phase-space picture like that shown in Fig. 2, if the attractor is of sufficiently simple topology, one can use methods which have been previously developed[1,4] to con-
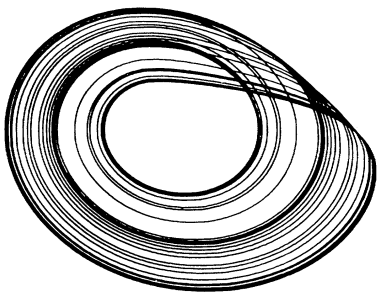

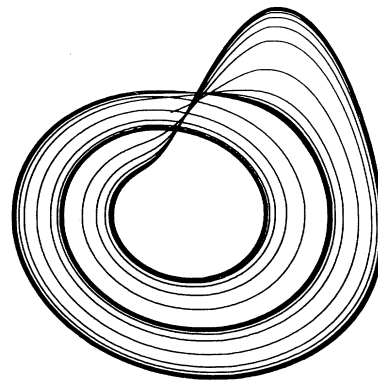FIG. 1. (x, y) projection of Rossler (Ref. 7).

FIG. 2. $(x, \dot{x})$ reconstruction from the time series.

struct a one-dimensional return map, and then from the return map one can obtain the positive characteristic exponent of the attractor. Roughly speaking, the procedure consists of making a cut along the attractor, coordinatizing it with the unit interval (0,1), and accumulating a return map by observing where successive passes of the trajectory through the cut occur. The result is a return map of the form $x(n+1) = f(x(n))$, and the positive characteristic exponent is found by computing

$$\lambda = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \ln \left| \frac{df}{dx} \right|_{x_i}$$

or alternately, by computing

$$\lambda = \int_0^1 P(x') \ln \left| \frac{df}{dx} \right|_{x'} dx'$$

if one knows (or has accumulated empirically) the equilibrium probability distribution $P(x)$. See Shaw[4] for more complete discussions of the computation of the characteristic exponents with use of this method.

Equations (1) are sufficiently simple that one can explicitly obtain a new set of three differential equations describing the dynamics of the state space comprised of a coordinate along with its first and second derivatives. Table I contains a comparison of the characteristic exponents for the original system [Eq. (1)], the transformed $(y, \dot{y}, \ddot{y})$ system, and the $(x, \dot{x})$ reconstructed return map, and shows very good agreement. The first two entries were obtained using the method of neighboring trajectories,[5,9] and the third entry was obtained using the return map method outlined above. The former method requires explicit knowledge of the dynamical equations, while the latter method depends on the dynamical system's attractor having sufficiently simple topology.

When trying to apply these reconstruction techniques to actual turbulence data, one of the first

TABLE I. Comparison of characteristic exponents from original (x,y,z) system, transformed $(y,\dot{y},\ddot{y})$ system, and construction of return map from $(x,\dot{x},\ddot{x})$ system.

	Characteristic exponent value
(x,y,z) system [Eq. (1)]	0.0677 ± 0.0005
$(y,\dot{y},\ddot{y})$ system [Eq. (2)]	0.0680 ± 0.0005
$(x,\dot{x})$ return map reconstruction	0.0677 ± 0.0001

questions will be exactly what dimension the system's attractor is. Note that the topological dimension of the attractor is directly related to the number of nonnegative characteristic exponents (see Bennetin, Galgani, and Strelcyn,[5] Shimada and Nagashima,[9] and Crutchfield[10]). A spectrum of all negative characteristic exponents implies a pointlike zero-dimensional attractor; one zero characteristic exponent with all others negative implies a one dimensional attractor; one positive and one zero characteristic exponent corresponds to the observation of folded-sheet-like structures making up the attractor; two positive characteristic exponents correspond to volumelike structures; and so on. The case of two zero characteristic exponents corresponds to a two-torus (two dimensional), but this should be distinguishable from a sheetlike chaotic attractor by the observation of two sharp incommensurate frequencies in the power spectrum. The dimension referred to above is the topological dimension of the attractor; we must realize that the Cantor-set structure typical of these objects implies a nonintegral fractal dimension[11] which can be expressed in terms of the characteristic exponents.[12] However, at any finite degree of resolution the observed topological dimension will be some integer value, though nonintegral dimension might be obtained by varying the resolution of the observation to see scaling in the structure of the attractor.

We now outline a procedure for determining the dimension of a smooth dynamical system. We begin with the idea that the "dimension" of a system being observed corresponds to the number of independent quantities needed to specify the state of the system at any given instant. Thus the observed dimension of an attractor is the number of independent quantities needed to specify a point on the attractor. For an attractor in an n-dimension phase space, we can discover the number of independent quantities needed for such a specification by slicing the attractor with $(n-1)$-dimension hypersheets defined by one coordinate being constant. The topological dimension of the attractor corresponds to the minimum number of sheets which, when intersected with each other and the attractor, will yield a countable number of points.[11]

If one chooses as phase-space coordinates the value of some variable along with time-delayed values of the same variable, this slicing of phase space can be accomplished by constructing conditional probability distributions. We define the

kth-order conditional probability distribution of a coordinate x, $P(x|x_1,x_2,\ldots;\tau)$, as the probability of observing the value x given that x_1 was observed time τ before, x_2 was observed time 2τ before, and so on. If we take τ to be small, the k conditions are equivalent to specification of the value of x at some time along with the value of all its derivatives up to order $k-1$. In fact, we must have $\tau \ll I/\Lambda$, where I is the degree of accuracy with which one can specify a state, and where Λ is the sum of all the positive-characteristic exponents, otherwise the information generating properties of the flow would randomize the samples with respect to each other.[4] In practice, it is easy to choose τ one or two orders of magnitude smaller than I/Λ. The dimensionality of the attractor is the number of conditions needed to yield an extremely sharp conditional probability distribution, in which case the system is determined by the conditions. These conditional probability distributions have been accumulated for the system given by Eq. (1), illustrated by the sequence in Fig. 3. We observe that the second-order conditional probability distribution is extremely sharp, implying that the attractor is two dimensional (sheets), which is indeed the observed structure. Other methods for determining the dimension of attractors will be reported elsewhere.[13]

The presence of observational noise in an experiment would be manifested in the increased width of the sharpest peaks obtainable in the sequence of probability distributions. For a noise level of δ, the width of the nth (sharp) probability distribution should be $\sim n\delta$. Thus for high-dimensional attractors, low-noise data is of paramount importance.

We have outlined techniques for reconstructing a phase-space picture from observing a single coordinate of any dynamical system. For systems which have only one positive-characteristic exponent along with sufficiently simple topology, we can obtain its value. We have also outlined a procedure for determining the dimensionality of an attractor from the observation of a single coordinate. All these techniques should be directly applicable to time series obtained from observing turbulence, as well as any other physical system, to construct a finite-dimensional phase-space picture of the system's attractor, provided such a low-dimensional structure exists. These ideas have recently been utilized by Roux et al.[14] to construct a phase-space picture of the chaotic attractor underlying chemical turbulence in the Belousof-Zhabotinsky reaction.
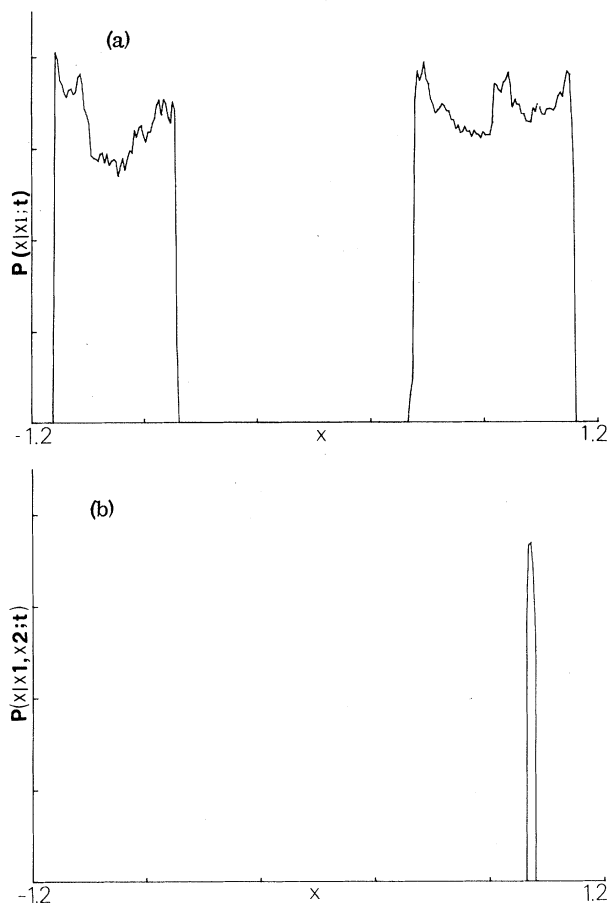

FIG. 3. Conditional-probability-distribution sequence for Eq. (1): (a) x vs $P(x|x_1;t)$; (b) x vs $P(x|x_1,x_2;t)$; where $x_1 = 0$, $x_2 = 0.495$, and $t = 0.2$ time units.

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Experimental Observation of the rf-Driven Current by the Lower-Hybrid Wave in a Tokamak

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It is observed that the waves launched from a phased array antenna of four waveguides couple effectively with electrons under the condition of $\omega_0/\omega_{lh}(0) \gtrsim 2.0$. This coupling generates a rf-driven current, rather than heating of the bulk electrons, and the current/rf-power ratio of 110 A/kW was obtained with a rf power of 125 kW radiated into a plasma which included appreciable suprathermal electrons.

A tokamak, which is the most successful device now on the road to controlled fusion, has the major disadvantage of pulsed operation because of a need to induce a toroidal current in the plasma. The application of rf to drive the current in steady-state tokamak reactors has been considered by a number of authors.[1-5] A method of producing continuous current carried by electrons in the tail of distribution function via quasilinear Landau damping of high–phase-velocity rf waves near the lower hybrid (LH) frequency has been proposed.[4,5] The linear and quasilinear Landau damping of slow electrostatic waves near LH frequency has been confirmed in a linear test device[6] and in the LH electron heating experiment on the tokamak (Doublet IIA).[7] These experiments provide a physical base for understanding the quasilinear Landau damping in the toroidal plasma with a relatively high electron temperature. Recently, the current generated by the unidirectional electron plasma waves has been observed in linear devices[8,9] and a toroidal device.[10] These experiments have been carried out in a plasma with a lower electron temperature, in which a transfer of momentum from LH waves to electrons via collisional absorption is significant.

In order to make effective coupling between the LH waves and electrons, it is necessary to avoid the deposition of the rf energy into ions resulting from the linear mode conversion and the excitation of parametric instabilities. The previous experiments on the rf ion heating indicated that for $\omega_0/\omega_{1h}(0) \gtrsim 1.6$ the ions did not interact with the rf waves and the parametric decay instabilities almost disappeared,[11,12] where ω_0 is the frequency of the applied rf field and $\omega_{1h}(0)$ is the LH frequency at the center of the plasma column. In this Letter, we report the experimental study on the coupling between the rf waves and electrons under the conditions of $\omega_0/\omega_{1h}(0) \gtrsim 2$ and the relatively high electron temperature in a tokamak.

The experiment, with a 750-MHz rf source, was performed in the JFT-2 (JAERI Fusion Torus) tokamak, which was a conventional tokamak with a major radius of $R_0 = 90$ cm and a minor radius of $a = 25$ cm. The experimental setup and the discharges were reported in detail,[13] and hence will be described only briefly here. In the present experiment, the following discharge was used as a magnetohydrodynamically stable operation; toroidal magnetic field $B_t = 14$ kG, plasma current $I_p = 30$ kA, mean line-of-sight electron density $\bar{n} \simeq 3 \times 10^{12}$ cm^{-3}, central electron temperature $T_{e0} \simeq (250 \text{ eV})/k$ and effective ionic charge Z_{eff} of 2–5. The working gas was deuterium. The waveguide array employed here consists of four independently driven waveguides mounted 1.5 cm away from the plasma edge, which is defined by