THE EFFECT OF FRICTION ON ELLIPTIC ORBITS

By D. G. Parkyn

1. *Introduction*

It forms an interesting and topical example in particle dynamics to investigate the effect of a small frictional perturbation on an elliptic orbit. In this paper the frictional force is assumed to be proportional to the square of the velocity and the eccentricity of the orbit is assumed to be small. Formulae are deduced for the rates of change of the major axis and eccentricity of the ellipse in terms of the rate of change of periodic time, numerical values being compared with the rather sparse information on Sputnik I.

2. If the resistance per unit mass is assumed to be kv^2, then, in the notation of Ramsey (*Dynamics*, Vol. II)

$$\dot{a} = -\frac{2ka^2v^3}{\mu}$$

and

$$\dot{e} = -\frac{2ka(1-e^2)}{e}\left(\frac{1}{r}-\frac{1}{a}\right)v.$$

Since $\dot{a}$ and $\dot{e}$ vary appreciably during a revolution, it is convenient to determine the changes in a and e, $(\Delta a, \Delta e)$ over one complete revolution. Thus, using the normal relations for elliptic orbits, we obtain

$$\Delta a = -2ka^2\int_0^{2\pi}\frac{(1+2e\cos\theta+e^2)^{3/2}}{(1+e\cos\theta)^2}\,d\theta,$$

$$\Delta e = -2ka(1-e^2)\int_0^{2\pi}\frac{(1+2e\cos\theta+e^2)^{1/2}\,(e+\cos\theta)}{(1+e\cos\theta)^2}\,d\theta.$$

Let

$$A = \int_0^{2\pi}\frac{(1+2e\cos\theta+e^2)^{1/2}}{(1+e\cos\theta)^2}\,d\theta$$

and

$$B = \int_0^{2\pi}\frac{(1+2e\cos\theta+e^2)^{1/2}\cos\theta}{(1+e\cos\theta)^2}\,d\theta,$$

then

$$\Delta a = -2ka^2\{(1+e^2)A+2eB\}$$

and

$$\Delta e = -2ka(1-e^2)\{eA+B\}.$$

To evaluate the integrals A and B expand them as power series in e and neglect terms above the square :

$$A = \int_0^{2\pi}\left\{1-e\cos\theta+\frac{e^2}{2}(1+\cos^2\theta)\right\}d\theta$$

$$= \frac{\pi}{2}(4+3e^2).$$

$$B = \int_0^{2\pi}\left\{1-e\cos\theta+\frac{e^2}{2}(1+\cos^2\theta)\right\}\cos\theta\,d\theta$$

$$= -\pi e.$$

Whence $\Delta a = -\pi ka^2(4+3e^2)$

and $\Delta e = -2\pi kae.$

Since $T = 2\pi\sqrt{\dfrac{a^3}{\mu}}$ we have that $\Delta a = n\dfrac{T\Delta T}{6\pi^2a^2}$, and hence $$\Delta e = \frac{\mu T \Delta T}{24\pi^2 a^3}(1 - \tfrac{3}{4}e^2).$$

3. *Numerical Results*

It is possible, assuming constant k, to integrate these relations and develop a, e and T as functions of the number of revolutions, n. However, the rates of change are so small that it seems adequate to assume a linear dependence upon n, especially as variation in k will eventually become predominant.

The available information on the initial orbit of Sputnik I is that its maximum height was 630 miles and that its apparent periodic time was 1·59 hours. This implies an actual periodic time of 1·696 hours, calculated on the number of revolutions attained on 13th November. Hence $a = 4{,}507$ miles and using the given apsidal distance we find $e = 0{\cdot}027$, and the other apsidal height as 384 miles. It has been stated that T is decreasing at the rate of 2·9 secs. per 24 hours.

From this we find
$$\Delta a = -0{\cdot}1103 \text{ miles}$$
and
$$\Delta e = -3{\cdot}04 \times 10^{-7}.$$

This would imply that the satellite would fall 120 miles in about 83 days, on the assumption of a constant density. The assumption that higher powers of e than the square may be neglected is amply justified for the given orbit, the only questionable assumption being the constancy of the density.

The statistically predicted density is of the order of 10^{-14} gms./c.c. at 100 miles and falls to 10^{-82} gms./c.c. at 560 miles, obviously inadequate to explain the stated rate of change of T. In fact, on the Newtonian resistance law—reasonably accurate for high speeds—we can estimate the required density as about 3×10^{-14} gms./c.c. One possible solution of the contradiction is in the presence of dust matter in the solar system, a reasonable supposition from observation of other galaxies. In this case one would expect a roughly constant density above the " statistical atmosphere ".

The resultant rate of fall then disagrees with the maximum apsidal distance published by the Russians on 11th November, 1957, of 503 miles, a fall of 127 miles in 39 days. In other words, the available information is inconsistent. The periodic time is probably correct and is capable of a simple check. The rate of decrease of periodic time may be compared with a Cambridge estimate of 1·8 secs. per 24 hrs., which would anyway give a slower rate of fall. An average height of about 500 miles is consistent with the periodic time, but it seems likely that the " maximum distances " are in error.

The validity of the original 630 miles would give the satellite a life of some 200 days, and of the second, a total life of 155 days. The average of these would allow a life of about 6 months—to mid-April, 1958.

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Additional Note

The paradoxical behaviour of Sputnik I, which travelled faster through the effect of friction, can also be demonstrated mathematically in the following elementary way.

Consider a particle describing a circular orbit of radius r_0 with speed v_0 under a central attraction κ/r^2. Then $v_0{}^2/r_0 = \kappa/r_0{}^2$ and $\kappa = r_0 v_0{}^2$. Now suppose that v and r change slowly with time under the influence of a tangential retardation λv_0. To a first approximation
$$v = r\dot{\theta} \quad \text{and} \quad \ddot{r} - r\dot{\theta}^2 = -\kappa/r^2$$
$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = -\lambda v_0.$$

Now suppose that after a short time t, $r = r_0(1+\alpha t)$, $v = v_0(1+\beta t)$, $\dot\theta = \dot\theta_0(1+\gamma t)$. Then the above equations give, neglecting t^2, and equating coefficients of t:

$$\beta = \alpha + \gamma$$
$$\alpha + 2\gamma = -2\alpha$$
$$\gamma + 2\alpha = -\lambda$$

from which $\alpha = -2\lambda$, $\beta = \lambda$, $\gamma = 3\lambda$.

Finally $r = r_0(1-2\lambda t)$, $v = v_0(1+\lambda t)$, $\dot\theta = \dot\theta_0(1+3\lambda t)$; showing that while r decreases with time, v and $\dot\theta$ both actually increase.

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RHOMBIC TRIACONTAHEDRA

BY JOHN D. EDE

The process of producing the facial planes of a polyhedron to form stellated polyhedra is well known. When applied to the icosahedron it leads to a series which is limited to eight solids if we make the condition that every face of each must be covered in forming the next of the series. If this condition is dropped a large number of combinations of the parts of these eight solids is possible. The total number of solids, limited by symmetry considerations, is 59, and they are all illustrated in *The 59 Icosahedra* by Coxeter, Du Val, Flather, and Petrie. The corresponding basic series of solids starting with the Rhombic Triacontahedron numbers 13, and the total number of combinations is very great. Details of the 13 solids are given in the table below.