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A Dynamic Systems Model of Cognitive and Language Growth

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In the first part of the article, a conceptual framework is sketched to define cognitive growth, including language growth, as a process of growth under limited resources. Important concepts are the process, level, and rate of growth; minimal structural growth level; carrying capacity and unutilized capacity for growth; and feedback delay. Second, a mathematical model of cognitive growth under limited resources is presented, with the conclusion that the most plausible model is a model of logistic growth with delayed feedback. Third, the model is transformed into a dynamic systems model based on the logistic growth equation. This model describes cognitive growth as a system of supportive and competitive interactions between growers. Models of normal logistic growth, U-shaped growth, bootstrap growth, and competitive growth are also presented. An overview is presented of forms of adaptation of resources (e.g., parental and tutorial assistance and support) to the growth characteristics of a cognitive or linguistic competence. Finally, the question of how the model can account for stages of growth is discussed.

The problem of quantitative increase of a capacity, skill, or knowledge base, a major concern for learning theories, is still largely unsolved in current structural theories of cognitive and language development. The theories address the problem of the emergence of new cognitive capabilities in terms of adding or deleting rules (or whatever the structural components are in the theory at issue) to or from a knowledge or procedural base. For instance, in a system of production rules that describes how the child solves the balance scale task (e.g., Siegler, 1983), a specific decision rule is either part of the rule system or not. In a transformational generative model of syntactic development, the child's grammar either contains a rule to change the subject–verb position in questions or not, but it does not contain rules such as "Change the subject–verb position arbitrarily in 27% of the cases." Put differently, the distinctions between developmentally different states of a cognitive structure occur in the form of discrete steps (see van Geert, 1987a, 1988c, for a more general account). However, the performance concerned does not change in the form of a sudden discrete leap corresponding to a state shift, but rather follows a gradual, sometimes irregular, increase (discussed later). Classical structural theories of cognitive development have accounted for such phenomena by introducing notions such as *décalage* or resistance of contents to assimilation by a specific operational structure (Smedslund, 1977) and distinctions such as between *competence* and *performance.*

In modern structural theories of cognitive development, information-processing models for example, the quantitative increase of cognitive capacities is of more central concern. For instance, some theories rely on the growth of working memory (e.g., Case, Marini, McKeough, Dennis, & Goldberg, 1986; Pascual-Leone, 1970) to explain structural changes, whereas others view specific forms of increase, namely S-shaped curves, as the quantitative analogon of a structural change (e.g., Fischer, 1980; Fischer & Canfield, 1986; Fischer & Pipp, 1984). The fact remains, however, that the quantitative aspect of cognitive development—referred to as *cognitive growth* in this article—is not the primary concern of structural models. They explain growth phenomena on the basis of transition mechanisms that are often peripheral to the structural core of the models. The problem of how to reconcile nongradual structural changes with gradual change in performance does not stand on its own. It is related to why in individuals the levels of development of various skills that allegedly refer to identical underlying competencies or structural bases are far less coherent than should be inferred from the underlying structural model (e.g., the discussion on the Piagetian *structure d'ensemble* concept, Piaget, 1972; Flavell, 1982). Although there is no general solution to the problem of quantitative increase in structural models, connectionist models of cognitive development seem to offer a way out of the impasse. The connections in an association network of cells change gradually, as does the performance based on the network state (e.g., Rumelhart & McClelland, 1987). A difficulty remains, however, in that one needs an explicit model of a network, one for learning the past tense of verbs for example, to generate a quantitative learning curve. In this article, I try to demonstrate that there is a general model of quantitative increase or decrease in cognitive development, namely *a dynamic systems model of logistic growth.* This model is intended to apply to all theories that subscribe to the idea that cognitive growth occurs under the constraint of limited resources.

The Concept of Cognitive Growth

A Definition of Cognitive Growth

I define *cognitive growth* as an autocatalytic quantitative increase in a growth variable following the emergence of a specific structural possibility in the cognitive system. Examples include the growth of vocabulary, the growth of subject–verb inversion in interrogative sentences, and the growth of the

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correct use of a strategy in solving fractions problems. There are three properties of cognitive growth. (a) The increase must be autocatalytic, that is, given no extrinsic impediments; growth is something that runs by itself. Any increase that amounts to mere addition from an outside source is not genuine growth. (b) It must be quantitative; growth is a property of a variable, the value of which can be expressed in terms of real whole or fractional numbers, such as the number of words or a percentage of correct applications. Growth could be used to describe qualitative changes such as structural development, but that is not the meaning addressed in this article. Of course, quantitative growth could result in qualitative changes in a system, but again this is not the major concern here. (c) Growth must follow a structural possibility of the cognitive system. That is, observable behavior, such as the use of words or grammatically correct interrogatives, should be considered to refer,[1] for instance, to the growth of vocabulary or to a subject–verb inversion rule only within the framework of a structural theory that considers words or interrogatives as the expression of a vocabulary or of a grammatical inversion rule. Although this may seem rather trivial, it implies that a growth model is always subsumed under a specific structural model, that is, a model providing specific cognitive interpretations for observable data (see van Geert, 1987a, 1990, for further discussion).

Growth Level and Growth Rate

I define a growth relation in set-theoretic terms as a relation with its domain a structural property and its range a field of applications. The growth relation is defined over the cardinality of this field of applications, that is, as a quantitative property. For example, the inversion rule is a structural property in some generative grammar model of language acquisition. Its range of application is those sentences to which the rule is actually applied. The cardinality of the range could be defined as the relative number of inversions in questions—that is, the number of inversions divided by the total number of questions during a time interval—or as the percentage of correct uses relative to all sentences in which, according to a correct grammar, the rule should be applied (e.g., Labov & Labov, 1978). The quantitative property of this range is called *growth level,* or *L*. For instance, the number of words actually understood by a child is the child's growth level of the structural property passive vocabulary, the percentage of correct questions with inversion is the growth level of the structural property inversion rule, and so on.

Thus, a growth relation *G* can be written as follows:

$$G:(S, t) \Rightarrow (L_{t1}, L_{t2}, L_{t3}, \ldots, L_{tn}), \tag{1}$$

where a growth relation *G* is a relation that maps a structural property *S*, defined over a time interval *t*, onto an ordered sequence of growth levels L_{t1}, L_{t2}, and so on; the growth levels correspond to the successive quantitative properties of the range of application of the structural property *S*. *G* defines a quantitative relationship between successive growth levels *L*, such as L_{t1}/L_{t2}, which takes the form of a regular increase or decrease, in other words, of a rate of change. Hence, *growth rate relation R* may be defined in set-theoretic terms as a relation that maps a growth level *L* onto another growth level, given a specific time interval between these two levels:

$$R:L_t \Rightarrow L_{t+i}. \tag{2a}$$

Given the quantitative nature of *L*, the relation *R* actually corresponds with a ratio *r*, for

$$r = L_{t+1}/L_t \text{ (e.g., } r = 0.2), \tag{2b}$$

where *r* is the growth rate.

Minimal Structural Growth Level and Growth-Onset Time

It follows from Equation 2b that there can be no initial growth level with a value of 0 (if $L_0 = 0$, multiplication by any *r* only leads to 0; i.e., the growth level remains 0). Because the initial state of growth is in principle the lowest possible growth level, and because this lowest possible level cannot be 0, it should be some arbitrarily small number (e.g., one word, or one correct application). This arbitrarily small number is the *minimal structural growth level* of a cognitive property or element. For instance, if making a subject–verb inversion in a question has become a structural possibility in a child's grammar, one may expect the child to use a minimal (and probably small) number of nonimitative questions using inversion. This is the rule's minimal structural growth level for this particular child. The *growth-onset time* is the moment at which a structurally minimal expression of a cognitive element emerges. For instance, the growth-onset time of a child's lexicon is theoretically the age at which the child has acquired his or her first real word. However, such minimal extension is not only hard to ascertain empirically, it is also likely that the minimal set is actually a few items (see the section on germinal phenomena).

Growth Processes Under Ecological Constraints

Given a specific cognitive theory, a Piagetian or an information-processing model for example, the structural elements discerned in this theory, such as skills, concepts, and rules, may be described metaphorically as cognitive species in a mental ecology: Each "species" occurs with a specific population (growth level) and relates to other "species," that is, to other structural elements (cf. Boulding, 1978). For instance, fast growth of the species *words* in a child probably will affect, positively or negatively, the growth in the species *grammatical knowledge.* This is so because one growth process may "feed" upon the other. For instance, the onset of grammatical growth may depend on the acquisition of some threshold number of words, in that skills necessary to learn new words contribute to the learning of grammatical rules, and rapid increases in grammatical knowledge consume part of the time and effort that might be used in building up the initial vocabulary (Dromi, 1986). I therefore

[1] *Refer* is used in the semantics sense; that is, observable behavior should refer to theoretical concepts, such as a grammatical rule. It is not implied that observables should refer to discursively interpreted mental states or structures, such as "rules in the head" or whatever; the reference relation does not imply the assignment of a specific ontological status to the theoretical concepts used such as "rules," "concepts," and so on.

compare the cognitive system of a developing individual to an evolving ecological system, which is not an ecosystem of animals and plants, but an ecosystem of cognitive "species" that take the form of rules, concepts, skills, and so forth.[2] The ecological metaphor is specified in five heuristic principles.

1. Given a specific structural model, the human cognitive system can be described as an ecosystem consisting of species (i.e., structural elements such as vocabulary, grammatical rules, problem-solving skills, and concepts) that entertain growth relationships with specific fields of application.
2. The elements engage in various types of functional relationships among one another, which are supportive (the growth in one supports the growth in another), competitive (the growth in one relates to the decline in another), or virtually neutral.
3. The elements show strongly dissimilar growth rates and growth-onset times.
4. The components compete for limited spatiotemporal, informational, energetic, and material resources.
5. (a) There exist more cognitive species (skills, knowledge items, rules, etc.) that can in principle be appropriated than actually will be appropriated by any particular person. (b) In principle, any cognitive species may occur with any possible growth level. (c) The set of cognitive species and respective growth levels characteristic of a person's cognitive system is the dynamic product of cognitive growth under limited resources.

The previous heuristic principles are reminiscent of those holding for biological ecological systems in general, and evolutionary systems in particular. For instance, the fifth principle is reminiscent of a principle in Darwinian theory, namely that the number of offspring exceeds the number of organisms that an environment is able to support long enough for the organism to reproduce (Gould, 1977). From this it follows that the adaptation of organisms to their environment increases over generations. Likewise, a heuristic claim is that learning under competition for limited resources favors species that can be learned more easily than others. Because learnability is dependent on the set of supporting cognitive resources that together form an individual's cognitive system, more easily learnable cognitive species (rules, skills, concepts, etc., that are more easily learnable in the individual's current cognitive system) tend over time to become more frequently represented in such a system than do less learnable ones (e.g., see Newport, 1982, for an application to language; van Geert, 1985).

In the framework of this ecological metaphor, I may speak about a *cognitive grower* and its *environment.* A cognitive grower can be any of the species in the mental ecology, or any structural element or component of a cognitive system to which the growth relation applies (thus, by *grower* I do not mean an individual child, but rather the child's vocabulary or the child's use of the inversion rule in questions). Trivially, a grower is a cognitive species that grows. The environment is the totality of supporting or competing resources upon which the grower feeds. Thus, as far as the nature of the cause, the magnitude, and time of the effect are concerned, I make no a priori distinction between subject-dependent resources (e.g., a child's mastery in dividing numbers will support the learning of rules for solving fractions) and external resources, such as amount of available models, tutorial support, and so on (see also Fogel & Thelen, 1987; Thelen, 1989).

The Nature of Limited Resources for Cognitive Growth

There exist many kinds of different resources that contribute to cognitive growth. I have just claimed that in principle no a priori distinction can be made among resources of different types (e.g., biological vs. environmental) in regard to their potential effect on growth (e.g., it is not so that in general, biological resources are more important than environmental ones, or the other way around). Decisions about the relative importance of resources depend on specific contexts and circumstances of growth. Nevertheless, for the sake of conveniently arranging an overview of the different resources, it could be handy to make a distinction between two dimensions. The first concerns the origin of resources and distinguishes between internal (in the subject) and external (outside the subject) ones; the second dimension deals with the nature of the resources, namely spatiotemporal, informational, energetic–motivational, and material resources. It is probably so that only in extreme cases will it be possible to distinguish the exact contribution of each of these types of resources. After all, the distinctions have no intrinsic theoretical meaning but primarily amount to a matter of "bookkeeping." The following overview is based on the two descriptive dimensions mentioned.

The concept of *internal spatial resources* refers to the limited amount of information one can deal with simultaneously (Kahneman, 1973; Miller, 1956) or to the limited range of one's working memory (Baddeley, 1976). The size of this mental capacity is believed to increase with age, either in the form of a literal increase or in the form of increasing efficiency of information processing (Case et al., 1986; Globerson, 1983; Pascual-Leone, 1970). However, the specific nature of the increase is still a much discussed issue (Case, 1984; Siegler, 1983). *Internal temporal resources* refer to the time on task that one is able or willing to invest in a specific cognitive activity, relative to the number of different cognitive activities carried out over a specific period. *Internal informational resources* consist of the knowledge and skills already present in the subject, which act as the internal learning or acquisition context for new skills and knowledge and which may either facilitate or impede the acquisition of specific new knowledge or skills. *Internal motivational/energetic resources* consist of the amount of energy, arousal, effort, activation, and so on invested in specific acquisition activities (e.g., Sanders, 1983). The energetic investment during specific information-processing activities may constitute a distinguishing property of normal and clinical groups in development (e.g., van der Meere, 1988). If energetic investment is defined as a content-specific variable, it can be called *motivation* (e.g., see Leontew's 1973 theory in which motivation plays a major developmental role). *Internal material resources* amount to the bodily outfit of a developing subject, for example, the availability of correctly working sensory and nervous systems.

[2] The metaphorical term *cognitive species* is similar to several terms introduced by scholars who have applied evolutionary analogies to the problem of the cultural transmission of knowledge and skills. They have proposed several terms to describe the units of such transmission; Dawkins (1976) used the term *meme* as the cognitive analogon to *gene;* Lumsden and Wilson (1981) used the term *culturgen;* see van Geert (1985) for an overview.

External spatiotemporal resources are the spatial and temporal degrees of freedom given to developing or learning subjects by their controlling environment. Caretakers and educators explicitly restrict the free-moving space and time of children, with the often implicit intention to structure this limited space in an optimally profitable way for the child. This principle is also inspired by the educator's need for a resource economy in the environment. Valsiner (1987) described this principle as the "zone of free movement." *External informational resources* primarily amount to the number, availability, and form of the items that could be assimilated by the developing and learning subject (e.g., the lexicon presented by the speaking environment, the specific ways in which the teaching environment makes information available to the learner). The third form of external resources is *energetic/motivational resources.* These are task-specific payoffs, such as the sort of reinforcement provided by the environment after performance of specific activities of the learner. *External material resources* are things like food and shelter, objects such as books and writing paper, and so on.

The availability, nature, and relationships of all these resources differ greatly among individuals and groups and also within individuals (e.g., temporal variations in the information given to a child or in the nature and amount of the energetic resources invested). However variable the resources may be, they are always limited, and as discussed later, this limited availability is one of the major formative forces in cognitive development.

The Concept of Carrying Capacity

At first sight it is very difficult, if not impossible, to characterize the amount of growth support that the cognitive environment provides to a specific grower in detail (e.g., vocabulary in a specific child) because this support characteristically amounts to various resources, many of which probably considerably vary over time. There is, however, a simple way to specify growth support. It is based on the following considerations. In principle, a grower might attain any possible maximal level if all available resources were invested in the grower concerned. However, there is competition for time, effort, information, and so on from other growers, and this limits any particular growth process. In fact, if too many resources were allocated to one specific grower, the whole cognitive system would become unstable and finally collapse. This is so because any individual grower depends strongly on the support of other growers and thus has to leave sufficient resources for the other growers to develop. Therefore, the highest possible level a grower may attain is automatically limited by the constraint of the long-term stability of the overall system.

In ecology, this limitation is associated with the concept of carrying capacity K of the system (De Sapio, 1976; Hofbauer & Sigmund, 1988). It expresses the long-term sum of resources supporting a specific grower over time, and it is specified in the form of *the maximal stable growth level of a particular grower in this specific cognitive environment* (i.e., stable under the condition that the present structure and amount of resources do not drastically change). This definition has important theoretical consequences. Because it is rather unusual to think in these terms in psychology, I return to a (pseudo-)biological example. Suppose that there is a hermetically closed cage that is populated with a couple of flies that are free of diseases. Suppose also that fixed amounts of food, water, oxygen, and so on are added per unit of time and that fixed amounts of waste products, carbon dioxide, and so on are withdrawn. What is added and withdrawn from the cage constitutes a multidimensional resource structure (in that each component, such as water, food, and waste, is independently variable). If the number of flies in the cage is the variable of focus, this multidimensional resource structure may be transformed into a one-dimensional measure, namely the maximal stable number of flies that cage can contain, given all the resources invested per unit of time. Thus, the multidimensional resource structure has not only been translated in a one-dimensional variable, it also has been translated in a variable that is qualitatively identical to the focus variable, namely a number of flies. Clearly, a change in the focus variable (e.g., number of very big flies) coincides with a change in the carrying capacity (the sum of resources must then be expressed in terms of the number of very big flies the cage may support). Another advantage of the concept of carrying capacity is that local variations in the amount of resources supplied do not necessarily affect the stability of the carrying capacity level itself. This is so for at least two reasons. First, if K is the expression of a multitude of resources, relative scarcity in one may be compensated within certain limits by relative abundance in another. Second, the variation (if not catastrophic) is a resource factor in itself: It is likely, for instance, that an irregular food supply with the same average as a regular one in another cage would have a slight negative effect on the number of flies the cage can sustain.

In cognitive development, one could say that the carrying capacity of a cognitive environment with regard to a cognitive grower such as vocabulary is a one-dimensional function of all the informational and tutorial support actually given to word learning, the time and energy spent in word learning, and the material support, such as books and toys. The one-dimensionality stems from the fact that given the investment of all these resources, there is a maximal level of vocabulary growth that can in principle be attained. This maximum level (e.g., 50,000 words during the life span in a literary culture or 350 words during the 1-word period for a bright child) is a quantitative measure of the sum of resources contributing to word learning during a given term. In summary, carrying capacity is a function that one-dimensionally expresses the sum of resources over time in terms of a maximal stable level a grower may attain given these resources.

External resource factors are specific in that they are in general more directly controllable than internal factors. An increase in any of the external factors that contribute to a specific carrying capacity (amount of food in the fly example; amount of parental help given in the example of word learning) would most likely lead to an increase in the carrying capacity, that is, in the highest possible stable level the grower at issue could achieve. Also likely, however, is that the effect of such increases would not pass beyond a ceiling level—that is, a level beyond which increase in external resource factors would no longer positively affect the carrying capacity. This upper limit forms the expression of the growth limitation inherent in the internal resource factors. For instance, young children will not learn abstract words—which they do not understand—no matter how much such words are used and explained by the environment. This upper limit plays an important role in Fischer's (1983a; Fischer & Silvern, 1985) skill theory. Most such upper limit K levels will change as a consequence of overall developmental changes in the cognitive environment. For instance, the child's growing understanding of multiple relations will make the learning of abstract words possible (Fischer, 1980) and will lead to a significant rise in the carrying capacity for vocabulary. In optimal circumstances this rise is partly caused by the increase in the internal resource factor (cognitive understanding) and partly by the increase in specific help that the environment presents following the change in the child. I discuss this mechanism further in the sections on bootstrap growth.

In summary, the carrying capacity is a one-dimensional variable closely linked to a specific one-dimensional growth variable, namely the growth level of a specific grower (e.g., words). It expresses the multidimensional structure of available resources in terms of the maximal stable level the grower at issue could achieve in the presence of these resources. Thus, it expresses resources in terms of the same dimension as the variable that is focused on, namely the level of a specific grower. Increase in external resources will in general lead to an upper limit in the carrying capacity, which is characteristic of intrinsic (but changeable) limitations in the internal resource factors.

The Concept of Unutilized Capacity for Growth

If K is the carrying capacity of a grower and L is its current growth level, the grower has to grow by K minus L items before it reaches its ceiling (that is, its growth limit arising from the limited resources in this particular cognitive environment). Because K is a measure of the resources available in a cognitive environment to a specific cognitive grower, $(K - L)$ is a measure of the amount of resources that can still be used to promote further growth. Thus, the function $(K - L)$ may be called the *unutilized capacity for growth,* denoted by U. As shown in a later section, it is a major component in logistic growth models.

The form of the growth relation associated with a growth process affected by a limited carrying capacity is as follows:

$$R{:}(L_t, K) \Rightarrow L_{t+i}. \qquad (3)$$

Given R, there is a ratio number r, such that $r = L_t/L_{t+i}$, and this ratio depends on the unutilized capacity for growth, $(K - L_t)$ or U_t.

Direct or Delayed Effects in Cognitive Growth Processes

If a teacher tells an adult nonnative student of English that sentences beginning with *Wh* require subject–verb inversion, the effect on the number of correct interrogative sentences uttered by the speaker is likely to increase immediately (it is assumed that the student did not know the rule). In this case, there is a direct (i.e., undelayed) effect of an increase in informational resources on the growth level of the inversion rule. Compare this situation with one in which a class gets a better English teacher than the pupils had previously. The introduction of the better teacher amounts to a sudden increase in informational resources and tutorial support. However, it will take some time before the effect of the better teaching is actually observable in the pupils' performance. Even in the example of the inversion rule, it is likely that a considerable amount of time will be needed before the student actually uses the rule consistently. That is, it takes a specific amount of time for the cognitive system to move from a state producing a performance p to a state producing a performance $(p + \Delta p)$. In this respect, the cognitive system is not different from other complex systems in nature. They are all dissipative systems (Brent, 1978; Stewart, 1989), and changing them means that a certain amount of inertia, friction, and resistance toward moving to a higher level of order must be overcome. This consumes time and energy.

The time lag between a growth state, that is, a specific growth level and its corresponding unutilized capacity for growth, and its effect on a later growth state is called *feedback delay,* denoted by f. The form of the growth relation of cognitive growth processes with feedback delay is as follows:

$$R{:}(L_t, K) \Rightarrow L_{t+f}. \qquad (4)$$

Note that the only difference from Equation 3 is that the index for the L to the right of the arrow has changed into $t + f$.

Feedback delay, as a content-specific expression of the inertia of the cognitive system, is not the same as learning time, although learning time contributes to feedback delay. For instance, at later stages of vocabulary learning, more words are learned during an equal time interval, and this is probably at least partly due to a considerable decrease in average learning time per word. Learning time is the average time needed for a word to move from an unlearned to a learned state (Greeno, 1974). Feedback delay, on the other hand, is the time lag between states—for instance, a present vocabulary level and a level that is $r \cdot (K - L)$ percent higher (discussed later). This time is not necessarily affected by changes in learning time.

Feedback delay is a measure of the inertia, friction, or resistance that the cognitive system must overcome to move from its present state to a more developed state. Although actual feedback delay depends on a myriad of factors, feedback delay as such is probably a constant property (i.e., constant given overall constancy of the cognitive system). This assumption is based on the general observation that complex systems tend to reduce the degrees of freedom of each of their components considerably, in that variability in many dimensions reduces to variability in a single dimension (Haken, 1987; Stewart, 1989; Thelen, 1989). Feedback delay may of course change, but such change will occur as a consequence of overall developmental changes in the cognitive system. That feedback delay can indeed be considered a constant should of course be demonstrated empirically. In this article, I show that a mathematical model using this constancy assumption yields good mathematical descriptions of regular and irregular empirical cognitive growth curves. Although feedback delay is different in different growers and at different times, the model presented herein is based on the simplifying assumption that feedback delays for all growers involved in a specific interaction are equal.

Germinal Phenomena and Allopatric Growth

In discussing the concept of growth rate, I have shown that an initial growth level must be a positive real number, however small. The concept of "minimal structural growth level" has been introduced to account for that fact. If the assumption is rejected that everything that can grow in cognition is innately present in some minimal, germinal form, then one must explain how the step from a nil state (growth level is zero) to a germinal state (growth level is an arbitrarily small positive number) can be made. This step cannot itself be a growth process. There are three logically discernible possibilities. First, the germinal state is innately given. Second, the germinal state has been inseminated from outside the developing subject; that is, it has been taught or imitated. Third, the germinal state has been constructed by the developing individual. One may question whether these logical possibilities also constitute psychologically relevant distinctions. With regard to the first possibility, presence in a germinal state actually refers to the innate nature of the concepts and strategies in question. Basic concepts in particular have an important germinal component. Examples include the notions of object (Spelke, 1985), humanity (Sylvester-Bradley, 1985), causality (Leslie, 1982), and number (Antell & Keating, 1983). This component is not the result of intellectual construction or teaching, nor is it qualitatively similar to the final state (van Geert, 1988a). For instance, the germinal state of the concept of *human being* probably amounts to a specific tendency on the part of the baby to pay attention to and interact with events that are in general typical of, but not exclusive to, animate objects (Sylvester-Bradley, 1985). The initial state of the concept of causality is probably a modular type of perception rule operating on mechanical causality events (Leslie, 1986). The actual onset of growth of these innate germinal states is probably timed by the growth of conditional or control variables.

The second possibility for making the step from a nil to a germinal state is by assimilating an externally presented model, specifically through imitation and demonstration or teaching. This process refers to the main source of intellectual growth as far as the transmission and appropriation of culture by every new generation are concerned. In teaching, the germinal form of a new grower is inseminated from outside, and its growth is carefully supported and controlled. However, imitation is a process that leads only to a germinal state of what has to be appropriated by the subject; that is, it is the starting point of a growth process. In this respect, imitation is similar to the effect of allegedly innate skills: What is innate is a specific starting point and possibility for learning and construction.

The third way in which a new grower can be initiated amounts to its autonomous construction by the subject himself or herself. That is, because there is neither an example that can be imitated nor any innate inclination, the subject discovers a new cognitive possibility. This is what probably occurs in creativity.

However, the three logical possibilities discerned—innately present, imitated, and self-constructed—refer only to potential germinal states, that is, to different types of starting points or initial states of cognitive growers. They do not make a difference with regard to the nature of the cognitive growth process itself, which always amounts to a process of construction. It is never so that a skill, form of knowledge, or whatever is innately given, or innately given in its complete form. Such form is always the result of a process of construction by the subject, regardless of the exact nature of the initial state or of the resources supplied.

The construction of new germinal forms in cognition is a major problem of development, originally discussed in Plato's *Menon,* which is concerned with the emergence of new forms out of old forms. This problem is still not satisfactorily tackled by existing theories of cognition (Thelen, 1989). The process of constructing new cognitive forms is probably similar to that in biology. Given a specific cognitive (or biological) structure, there exists a limited domain of degrees of freedom for constructing new forms (Ho & Saunders, 1984; Saunders, 1984). The construction of new forms is an intrinsic possibility of a cognitive system, in that its reproduction over time or its maintenance is vulnerable to random perturbation (mutation) and to imported models (imitation; Fogel & Thelen, 1987; Siegler, 1984). In some cases, these unintended mutations of some existing cognitive capacity are selected and supported by the external environment. A good example is the early growth of words, based on meanings given by the adults to protomeaningful acoustic productions in a baby (e.g., see Jakobson, 1960, on the growth of *mommy–daddy* words). In general, however, newly emerging forms have to compete with those that already exist, and although in the long run the new forms will turn out to be more powerful than existing ones (e.g., operational as opposed to preoperational thinking), they are definitely much less powerful at the time they emerge in a germinal form. In evolutionary biology, a comparable problem occurs in explaining the emergence of new species, namely the problem of cladogenesis (Gottlieb, 1984). It is often solved by using the concept of *allopatric growth* or *allopatric speciation* (Mayr, 1976; Simpson, 1983). Allopatric speciation is rapid evolutionary change in a geographically separated (i.e., frontier) part of the original species population. Because the separated part occupies its own small habitat, relatively isolated from the mainland, it can change under relatively safe circumstances, with little or no competition from the main species. Later, the altered species form, if better adapted to circumstances that might have changed in the meantime, may take over the habitat of the original main population. Applied to cognitive development, allopatric growth means that a new capacity, rule, and so on may be constructed by random variation, selection, imitation, and so on. This may occur in a relatively isolated and uncompetitive subfield of the field of application of an already established capacity, rule, or whatever. Allopatric cognitive growth is a natural phenomenon, because almost all fields of application of a rule or production system break down into subfields.

These subfields are characterized by differences in cognitive complexity, difficulty, specific domain of application, and so on. A particularly clear example is offered in Klausmeier and Allen's (1978) longitudinal study of concept development during the school years. The authors distinguished four conceptual rule systems that form a developmental sequence, namely concrete, identity, classificatory, and formal levels. They observed that conceptual development is not equal for all concepts at all levels. For instance, there is a natural *décalage* between object, geometric, and abstract concepts and between concepts within each domain as far as speed and ease of development are concerned. A very difficult task would be to construct a new concept rule system (e.g., a classificatory level) for the whole do-

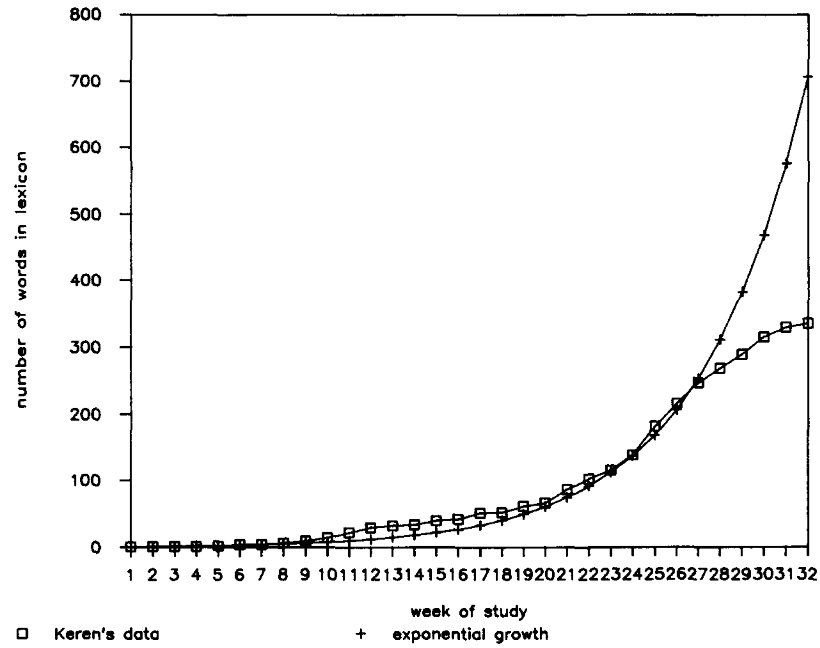

Figure 1. Keren's cumulative lexicon during the one-word stage compared with an exponential growth curve, where growth rate $r = 0.2$ per week. (Adapted from "The One-Word Period as a Stage in Language Development: Quantitative and Qualitative Accounts," p. 229, by E. Dromi, 1986, in I. Levin, *Stage and Structure: Reopening the Debate,* Norwood, NJ: Ablex. Copyright 1986 by Ablex. Adapted by permission.)

main of concepts at once, but this task would be relatively easy for limited domains, such as a particular concept from a particular class (e.g., the concept of *cutting tool* from the object class). By the time a conceptual strategy in a more difficult concept domain has reached the limit set by its carrying capacity, a more advanced strategy is likely to be ready in a relatively isolated, easier subdomain (e.g., a specific concept). The latter may then be adapted to the requirements of the more complex conceptual domain with relative ease. Thus, instead of being some sort of nuisance, as for instance in Piaget's model, *décalages* are the key to cognitive development, in that they create opportunities for allopatric growth of new cognitive forms. *Décalages* may occur because fields of application of cognitive rules, concepts, and so on have a laminar structure; that is, they tend to fall apart into (weakly) independent subfields. Although the problem of the creation of new cognitive forms is indeed a central problem that requires much further scrutiny, in this article I confine myself to discussing growth following the installation of a germinal growth form.

Toward a Mathematical Model of Cognitive and Language Growth

Exponential Growth Without Feedback Delay

I start from the simplest possible assumption, namely that, for all practical purposes, during a given interval t, a specific cognitive growth process is not intrinsically restricted and that no feedback delay occurs. Given the growth rate relations (Equation 2),

$$\Delta L/\Delta t = L \cdot r, \tag{5}$$

where Δ is the symbol for difference. Because there is no feedback delay, Equation 5 may be differentiated:

$$L_t = L_0 \cdot e^{r \cdot t}, \tag{6}$$

where e is the base of the natural logarithm, which is the classical formula for exponential growth. To test whether an empirical growth curve is actually modeled according to this equation, its growth rate may be computed with a formula inferred from Equation 6, namely

$$[\ln(L_2) - \ln(L_1)]/t. \tag{7}$$

The empirical growth rate results from taking the difference between the natural logarithm of two consecutive growth levels and dividing this number by the number of units of time; L_1 and L_2 represent the size of a growth variable at two different moments. Possible examples include the lexicon or the number of problems solved in a standard test.[3]

Let me first try to apply this concept of growth rate to an empirical example. Dromi (1986) studied the growth of the lexicon in a girl, Keren, between the ages of 10 and 17 months. Figure 1 shows the cumulative lexicon (see also Gillis, 1984, and K. Nelson, 1985, for comparable data).[4] In view of the intensive data-gathering procedure normally used in these $n = 1$ language studies, the measurement error may be considered to be rather low.

[3] This formula is identical to Haldane's formula for relative speed of change in evolution (Simpson, 1983).

[4] The parameters and equations necessary to reconstruct the present theoretical curve are described in the text. Details are described in the Appendix.

Growth rate may be computed by taking two points on the growth curve, for example, at t_{27} (where the number of words is 227) and t_1 (where the number of words is 1). Computed growth rate is about 0.2 per week (i.e., about 0.88 per month, which is very high). If the exponential and the empirical growth curves are compared, one sees that both run closely together until about Week 27. At this point, the empirical curve levels off, but the exponential curve continues to increase. It is interesting that the sudden drop relative to the exponential curve coincides with the onset of syntactic development, as shown by the child's use of multiword sentences (Dromi, 1986). My first tentative conclusion from this example on vocabulary growth is that initial growth rate—in this case, vocabulary growth during the 1-word stage—should be very high, or at least much higher than later growth rate. If it were low, it would take too much time to build up a critical mass in the domain at issue. Thus, initial growth rate should be quite high. The second conclusion is that high initial growth rate should drop very quickly after some critical mass has been achieved, simply because a continuing high growth rate would lead to exhaustion of all available resources, for example, learning time. For instance, if the initial growth rate of 0.88 per month in early lexical development were to continue, by the 16th month of learning the child would have to assimilate 17 items every minute, nights included, to keep up with the growth rate. This is physically and psychologically impossible. Consequently, a model that does not explicitly take resource limitations into account seems untenable.

Logistic Growth Without Feedback Delay

In the previous example, vocabulary growth slowed down at the end of the one-word stage for the child concerned. Why or when exactly this deceleration occurred does not matter, but that it occurs is necessary, because otherwise the exponential growth would rise too rapidly. The deceleration could be the effect of an underlying growth program's putting the brakes on Keren's word learning and accelerating her syntax learning. However, slight delays in the onset of the preprogrammed braking action would provide an exponential grower that is only slightly in advance of others at the beginning of the growth process, with the opportunity to rise to an extremely high level. The price of this exponential eruption would be the consumption of almost all the available resources, and this would seriously jeopardize the growth of potential supportive skills and knowledge. However, the model includes the assumption that the collection of supportive and competitive relations among cognitive growers in a single subject sets intrinsic and specific growth limits for any individual grower in the form of a specific carrying capacity K. For instance, there is an intrinsic limit to the number of words that are accessible to the child at any given moment (MacNamara, 1982; K. Nelson, 1985). This accessibility is based on various factors, such as the average word exposition time per day, the frequency and number of words used by the more competent speakers in the child's environment, and, in particular, the cognitive accessibility of words. Imagine a child growing up in a family of dog breeders with tax problems: Although both words occur very frequently, the young child will easily learn the word *dog* but probably not *taxes*.[5]

Let K be the (temporary) carrying capacity resulting from the limited-resource factors mentioned earlier, and let L be the number of words already assimilated in the vocabulary. At any moment, the unutilized opportunity for vocabulary growth is $(K - L)$. From the definition of growth under restricted carrying capacity (see Equation 3), one may infer that growth is a function of the number of items already acquired relative to the maximum of learnable items; that is,

$$\Delta L/\Delta t = r \cdot L \cdot U \quad (8a)$$

for

$$U = K - L. \quad (8b)$$

Carrying capacity and the unutilized opportunity for growth inferred from it may take different psychologically operational forms. For instance, if a minimal number of encounters with an item is required in order for it to be assimilated, then the more items one knows the lower the probability that an encountered item will be unknown, and therefore the lower the probability that a contribution to item growth will be made. Another possibility is that easily learned items are learned first (i.e., they are the first to move from an unlearned to a learned state), such that learning rate becomes slower as the number of unlearned items grows smaller. The more difficult an item, the longer it will remain in the unlearned item set. Finally, U may consist of tutorial assistance that decreases as the level of mastery of the tutee increases. Any combination of such factors is also possible.

In line with classical approaches to logistic growth in biological sciences, I shall (provisionally) assume that there is no feedback delay, that Δt approaches 0. Thus, Equation 8a may be differentiated to find the classical *logistic growth function* (De Sapio, 1976):

$$L_t = K/(1 + c \cdot e^{-K \cdot r \cdot t}) \quad (9)$$

for

$$c = K/L_0 - 1, \quad (10)$$

meaning that the growth level at time t is a function of a fraction, where the numerator is the carrying capacity and the denominator contains the product of a constant c (Equation 10) with the exponential function of the negative product of the carrying capacity K, the growth rate r, and time t). If K is set arbitrarily to 1 and L is expressed as a fraction of 1, logistic growth rate in a given empirical growth process can be computed according to the formula

$$r = -\ln[(1 - L)/(L \cdot c)]/t. \quad (11)$$

This equation is applied to the data from Dromi's (1986) study. Provided the drop in lexical growth rate at the end of the one-word stage indeed refers to an upper limit of accessible words, this limit may be estimated at 350 and then set arbitrarily to 1. Then the number of words at Week 32 ($n = 335$) is expressed as a fraction of 1. Equation 11 yields a weekly growth

[5] It is very likely that cognitive and physical accessibilities are also subject to growth. However, particularly in the first phase of word acquisition, one may expect that vocabulary growth rate will be much higher than the growth rate of the accessibility threshold, and in general, of the carrying capacity for vocabulary.

rate of about 0.28, which is very high. Figure 2 shows that the logistic curve, although roughly similar to the empirical curve in its general S shape, differs considerably from the latter in terms of its slope. This lack of empirical fit implies that the chosen growth model—logistic with feedback delay equal to 0—is inadequate for the present data. However, consideration must be given to whether there is a form of cognitive growth that can be modeled after the present logistic/no-delay model.

Fischer and Pipp (1984; see also Fischer & Canfield, 1986) claimed that cognitive growth phenomena take place in the form of S-shaped growth spurts (e.g., see the growth in correct application of the concept of *sweetness* in Strauss & Stavy, 1982). They further explained that such spurts are seldom found because standard testing conditions provide a distorted image of real cognitive growth. To reveal such growth curves, testing practices are needed in which the subjects are given feedback to their answers and additional support (see Figure 3). If the growth rate is computed according to Equation 11, there is a yearly growth rate of 1.26. The logistic curve based on this growth rate approximates the empirical curve only roughly (Figure 3). In a later section of this article, I present a dynamic systems application of the logistic curve and show how a much better fit with irregular curves may be achieved.

Restricted Growth Without Feedback Delay

In the previous section the assumption was tested that growth is a nondelayed function of L and U (i.e., of a present growth level and the resulting unutilized capacity for growth, $K - L$). There was only a rough qualitative fit to the data, in that both the empirical and the theoretical curve had an S shape. Before studying the effect of feedback delay on growth, I test the assumption that growth is only a function of U (i.e., $K - L$) and not of U and L. The equation for the curve is as follows:

$$\Delta L/\Delta t = r \cdot (K - L). \tag{12}$$

If there is no feedback delay, Equation 12 may be differentiated to yield

$$L_t = K - (K - L_0) \cdot e^{-r \cdot t}, \tag{13}$$

and r can be computed as follows:

$$r = \ln[(K - L_i)/(K - L_{i+t})]/t. \tag{14}$$

An example of this type of growth is offered by Klausmeier and Allen's (1978) longitudinal study of concept development during the school years. Four levels of processing conceptual information were distinguished. Children learn to solve increasingly complex problems at each level for each specific concept (e.g., the classificatory level of the concept *noun*). Most of the growth curves reported in Klausmeier and Allen's study take the form depicted in Figure 4. These growth curves are typical of restricted growth, that is, growth that is solely determined by the unutilized opportunity for growth U, which is the difference between the growth level already acquired and the maximal growth level K. For the empirical growth curve shown in Figure 4, Equation 14 yields an average growth rate of 0.28 (for $K = 1$). The resulting computed growth curve follows the empirical one very closely. Growth equations (12 and 13) imply that the conceptual growth measured by Klausmeier and Allen is entirely determined by negative evidence, that is, by the nature and amount of the problems the child is not yet able to solve. This follows from the fact that in these equations growth is not determined by L—that is, the knowledge the child already has—but by $K - L$—that is, the knowledge the child has not yet achieved. There is no way for the child to know or be confronted with this unknown knowledge other than in the form of the errors the child makes, the problems he or she is confronted

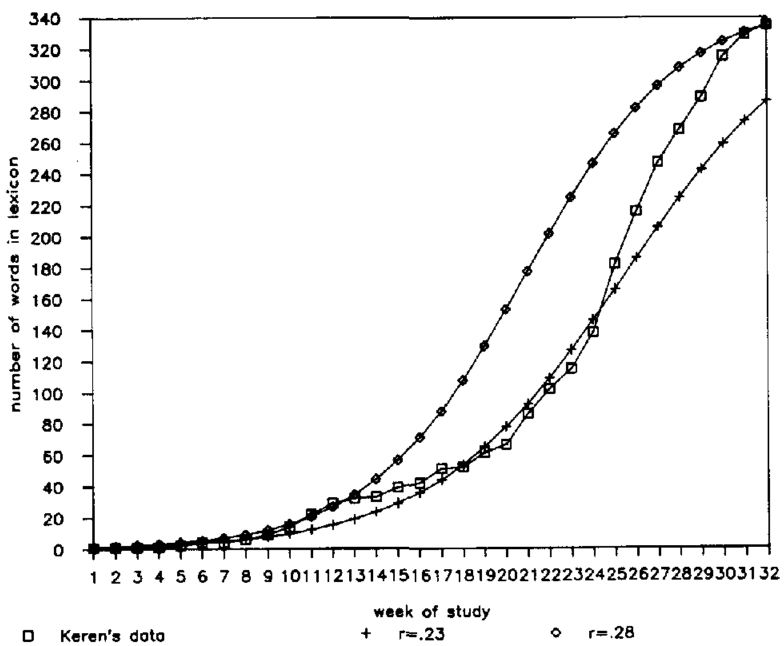

Figure 2. Empirical curve of Keren's vocabulary growth compared with logistic curves, where $K = 350$ words and growth rates are 0.23 and 0.28.

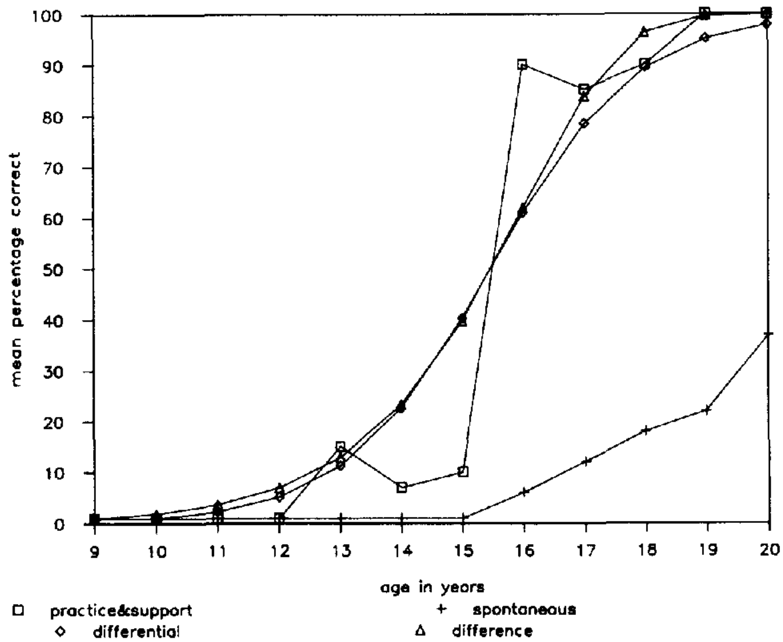

Figure 3. The growth of arithmetic problem solving under practice-and-support and spontaneous conditions compared with theoretical curves (difference and differential form). (Adapted from "Processes of Cognitive Development: Optimal Level and Skill Acquisition," p. 57, by K. W. Fischer and S. L. Pipp, 1984, in R. J. Sternberg, *Mechanisms of Cognitive Development*, New York: W. H. Freeman. Copyright 1984 by W. H. Freeman. Adapted by permission.)

with that he or she is not yet able to solve, the corrections made by a tutor, and so on. The form of the Klausmeier and Allen curves is very similar to that of the classical learning curves (Hilgard & Bower, 1966). It is not too difficult to fit these learning curves by applying equations that define learning as the negative growth of what the learner does not yet know (see van Geert, in press). The major problem with Equation 13, however, is that it does not model S-shaped curves (only the upper part of the S shape) that are characteristic of many forms of cognitive growth. Thus, it appears that the assumption un-

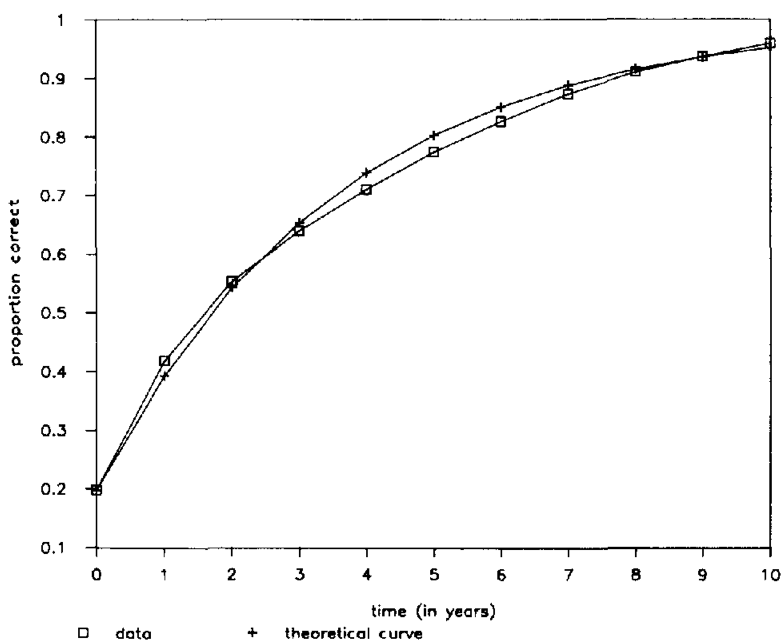

Figure 4. Growth of noun-concept problem solving at the classificatory level compared with curve for restricted growth, where $r = 0.28$. (Adapted from *Cognitive Development of Children and Youth: A Longitudinal Study,* p. 12, by H. J. Klausmeier and P. S. Allen, 1978, Madison: Board of Regents of the University of Wisconsin System. Copyright 1978 by the Board of Regents of the University of Wisconsin System. Adapted by permission.)

derlying the logistic equation (Equation 9) is basically correct and that Equation 13 models only a subset of growth forms. I next test the assumption that a better fit can be achieved by introducing feedback delay.

Feedback Delay and Approximative, Oscillatory, and Chaotic Growth

The delayed-feedback hypothesis can be covered in a simple mathematical model, namely one that does not use differential operators, as with the growth equations discussed earlier, but instead uses difference operators (Burghes & Wood, 1985). The assumptions are a growth level L_t (e.g., the number of words a child knows at his or her second birthday) and a carrying capacity K (the maximal numbers of words the child could acquire and maintain, given the present conditions of internal and external cognitive support). The state of the growth level may be inferred at a later moment, where *later* means after feedback delay time has elapsed, according to the growth form (Equation 3) as follows:

$$L_{t+f} = (1 + r') \cdot L_t. \quad (15)$$

Also, the rate r' is a function of the unutilized opportunity for growth. However, if growth rate is to be treated as a dimensionless variable, the value of which is not dependent on the absolute magnitude of the variables on which it operates, then r' should be assumed to be a function of the relative unused opportunity for growth and a constant intrinsic growth rate r (Hofbauer & Sigmund, 1988):

$$r' = r \cdot (K - L_t)/K. \quad (16)$$

Combining Equations 15 and 16 yields the *difference equation for logistic growth,* that is, the equation for logistic growth with feedback delay, namely

$$L_{t+f} = (1 + r) \cdot L_t - r \cdot L_t^2/K. \quad (17a)$$

This iterative equation specifies any point on a growth curve as a function of a point that occurs a temporal interval f earlier. Given an initial growth level, it is easy to generate a growth curve as a set of points at a mutual time distance of f (in another section, I discuss how to infer intermediary points).

An interesting notational variant of Equation 17a is

$$L_{t+f} = (1 + r - a \cdot L_t) \cdot L_t \quad (17b)$$

for

$$a = r/K \quad \text{and} \quad K = r/a \quad (17c)$$

(in Equation 17b, a is used as a braking parameter).

Equation 17a has a stable solution, in that L no longer changes. To find this stable solution, the variable part of Equation 17a is set to 0; that is,

$$r \cdot L_t = r \cdot L_t^2/K. \quad (18)$$

It follows that Equation 17a has a stable solution when

$$L_t = K. \quad (19)$$

The stable solution is an important property of the logistic growth curve, as shown later. Given an empirical curve, one could infer the growth rate r by taking two points at a distance f from one another according to the following equation:

$$r_t = [(L_{t+f}/L_t) - 1)]/[1 - (L_t/K)]. \quad (20)$$

Because f is unknown, any two consecutive points on an empirical curve can be used to compute a corresponding r. It can be shown, however, that the overall fit of the theoretical growth curve increases as the estimated time interval approaches the real feedback delay (see Figure 5).

Actually, time does not appear as a real variable in Equation 17a (in contrast to time in the ordinary logistic growth Equation 9). Time is nothing but an index variable. The unit of time is the feedback delay. An empirical interpretation of the real length of f may follow from curve fitting: Given an empirical curve of which the total growth time t is known, feedback delay equals t divided by the number of iterations used to reconstruct the empirical curve mathematically.

The previous iterative growth equation has proved its validity in a wide range of applications, such as meteorology, population dynamics, economics, and fluid dynamics (Abraham, 1987; Abraham & Shaw, 1987; Garfinkel, 1987; Gleick, 1987; Hofbauer & Sigmund, 1988). Peitgen and Richter (1986) termed these dynamics *Verhulst dynamics,* named after a Belgian 19th-century mathematician and population researcher. These dynamics have a number of interesting and unexpected properties, despite the very simple character of the basic equation (Schuster, 1988).

I next explore whether this simple model is indeed capable of giving a simple mathematical explanation for some of the growth forms found empirically. I have shown that the ordinary, differential logistic equation sharply overestimated the first half of the growth curve in Dromi's (1986) study and underestimated the second half. With different feedback delays, for instance 1 and 2 weeks, Equation 20 can be used to find the corresponding rs. The best overall approximation seems to be one with a feedback delay of 2 weeks and $r = 0.71$ (see Figure 6). The major argument for taking a feedback delay of 2 weeks is that it provides a better fitting curve than a feedback delay of 1 week. A feedback delay of 2 weeks is not necessarily the most optimal solution (although better than 1 week), but it is easy to test, because the data are based on weekly measurements. An additional empirical argument is that 2 weeks is the average time for a word to stay in an underextended state (Dromi, 1986). A still better fit can be obtained by considering the empirical curve as a two-step process. The first step seems to be an initial growth period stabilizing at about 25 words (Week 12). Then a secondary growth period follows, starting at the 25-word level and stabilizing at about 350 words. A good-fitting curve for the second substage has f = 1 week and $r = 0.35$ (see Figure 7). The corresponding undelayed feedback curve either strongly overestimates the initial growth speed or strongly underestimates the growth speed toward the steady state (Figure 7). The two-substage hypothesis is supported by the fact that the development of word meaning in Keren's vocabulary proceeded in two different stages, one in which semantic extensions of newly acquired words were unpredictable and a second in which those extensions were regular and closely followed the adult meanings (Dromi, 1986). These two types of meaning acquisition were clearly differentiated only at about Week 19,

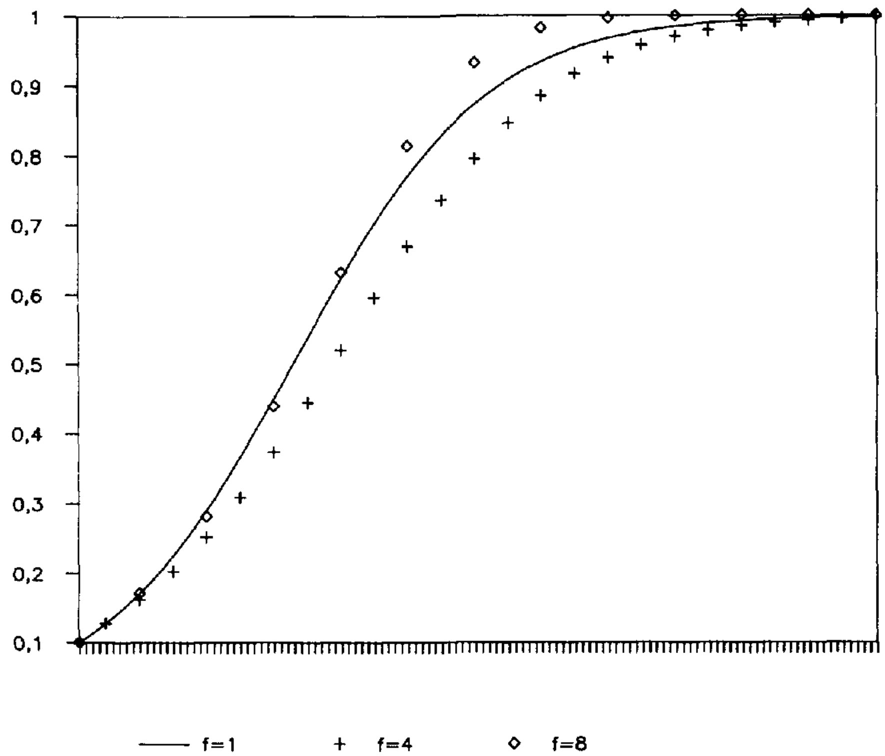

Figure 5. The length of feedback delay defines a unique curve: Differences in growth rate r cannot compensate for different feedback delay (f = 1 vs. 4 and 8).

although Week 12 was considered the point of separation. This delay might be caused by the fact that the observable overt expression of a growth phenomenon is delayed relative to its actual growth onset or by the fact that the two substages of vocabulary growth overlap. A general explanation for the possible two-substage character might be found in the potential differences in cognitive requirements and contents of initial words (e.g., those might be more directly related to sensorimotor discoveries; Brown, 1973; Gopnik, 1984). A related explanation refers to the child's discovery of the "naming insight" (K. Nelson, 1985). Such assumptions of course remain to be tested empirically. In the next part of this article, however, I discuss a dynamics model that generates stepwise growth, comparable with the stepwise growth in Dromi's data, without referring to ad hoc structural explanations. Finally, please note that the growth rates and feedback delays found in this example are typical of Dromi's subject and do not constitute general or universal parameters of vocabulary growth. Later, I discuss other data (Corrigan, 1983) with different lexical growth curves. It is likely that these parameters vary among subjects and that upper and lower boundaries of these parameters define "normality."

An interesting feature of the iterative logistic growth equation is that it is capable of describing nonlinear growth phenomena and even near-chaotic growth (Peitgen & Richter, 1986; Schuster, 1988). For $r < 1$, the curve has the characteristic S shape or sigmoidal form seen in the vocabulary data. This may be called *asymptotic growth.* For $1 < r < 2$, the curve shoots above the stable solution level, which is the carrying capacity (see Equations 18 and 19), then drops and moves on in a decreasing vibration, aiming at the level of the carrying capacity (Figure 8, bottom curve). This may be called *approximate growth.* The carrying capacity, which is the stable solution of the logistic growth equation, is a so-called point attractor for all growth processes where $r < 2$. For $2 < r < 2.57$, the growth process has 2^n attractors, and growth takes the form of a periodic oscillation (Figure 8, middle curve), which may be called *oscillatory growth.* Above 2.57, the process loses its periodicity and moves into chaos, or *near-chaotic growth* (Figure 8, top curve). The previous processes provide good illustrations of the principle of phase shifts following boundary transitions of a gradually and monotonically changing control parameter (Fogel & Thelen, 1987). Linear increases in the growth parameter lead growth processes over several sharply distinguished types (asymptotic, approximative, etc.).

Learning and Forgetting Under Limited Resources

In the previous section, I discussed cognitive growth as a form of increase, expressed by the growth rate r. Obviously, however, there is not only increase, in the form of acquiring or learning words, skills, and so on, but also forgetting, the loss of proficiency, and so on. Thus, a forgetting factor needs to be included in the logistic growth equation. Let m' be a rate of forgetting, loss of competence, and so on. In accordance with the basic growth form and Equation 15,

$$L_{t+f} = (1 + r' - m') \cdot L_t. \qquad (15')$$

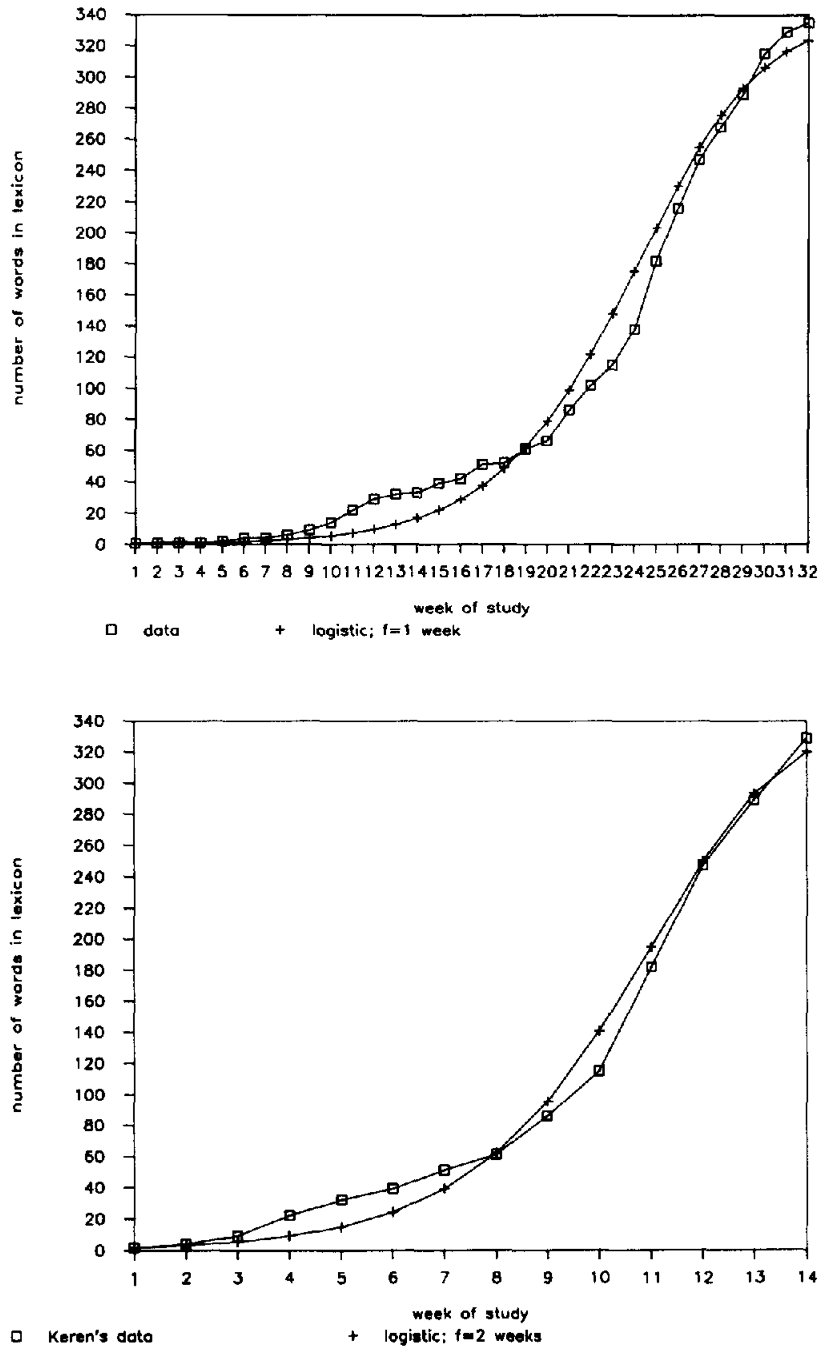

Figure 6. Empirical curve of vocabulary growth (Dromi, 1986) compared with mathematical curves based on the difference form of the logistic equation for feedback delays of 1 and 2 weeks. (Adapted from "The One-Word Period as a Stage in Language Development: Quantitative and Qualitative Accounts," p. 229, by E. Dromi, 1986, in I. Levin, *Stage and Structure: Reopening the Debate,* Norwood, NJ: Ablex. Copyright 1986 by Ablex. Adapted by permission.)

One may assume that m' is the complement of r'. Thus, whereas r' is dependent on the unutilized opportunity for growth U, m' is related to U's complement. Because $U = (K - L)/K$, its complement is L/K:

$$m' = m \cdot L_t/K. \qquad (21)$$

Substituting Equations 16 and 21 in Equation 15′ yields the *difference equation for logistic growth and forgetting:*

$$L_{t+f} = (1 + r) \cdot L_t - (r + m) \cdot L_t^2/K. \qquad (17a')$$

This equation has a stable solution when the sum of the variable parts is 0; that is,

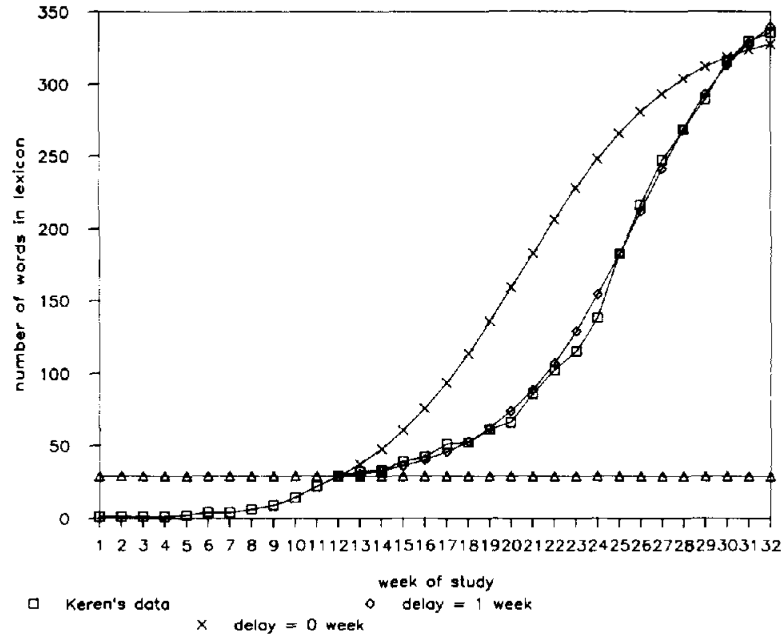

Figure 7. Mathematical curves of vocabulary growth in second substage of one-word stage: Differential logistic curve (f = 0) compared with difference form (f = 1 week, $r = 0.35$) and empirical curve. (The line of triangles marks the assumed base level for the second substage.)

$$r \cdot L_t = (r + m) \cdot L_t^2 / K. \qquad (18')$$

From Equation 18′ it follows that 17a′ has a stable solution when

$$L_t = K \cdot r/(r + m). \qquad (22)$$

This means that if a learning-and-forgetting curve grows toward a stable level, it grows toward a fraction $r/(r + m)$ of its carrying capacity. It may now be questioned whether there exists a growth rate r'', producing a growth process that is identical to a growth process produced by a growth rate r and a forgetting rate m, with a stable solution that is described by Equation 22. That is, what is the value of r'' if

$$(1 + r'') \cdot L_t - r'' \cdot L_t^2 \cdot (r + m)/(K \cdot r)$$
$$= (1 + r) \cdot L_t - (r + m) \cdot L_t^2/K? \qquad (23a)$$

The answer from solving the above equation is that

$$r'' = r. \qquad (23b)$$

Put differently, the logistic curve for growth and forgetting is identical to the curve for growth only, provided that the carrying capacity of the latter is set to

$$K'' = K \cdot r/(r + m). \qquad (23c)$$

This means that the effect of forgetting or loss of proficiency merely consists of lowering the carrying capacity, in comparison with the case in which no or less forgetting occurs, whereas the growth rate r stays the same. That is, in Equation 17a, the rate of growth is the net result of actual learning and forgetting processes, and the carrying capacity K is a function of pure learning and pure forgetting; that is, K is lower the higher the rate of forgetting. If forgetting were not a resource-dependent function—that is, if Equation 21 would not hold—there would be an entirely different growth curve. However, if learning, the sheer cognitive growth increase, is a resource-dependent function, it would be very surprising if its complement, forgetting, would be resource independent.

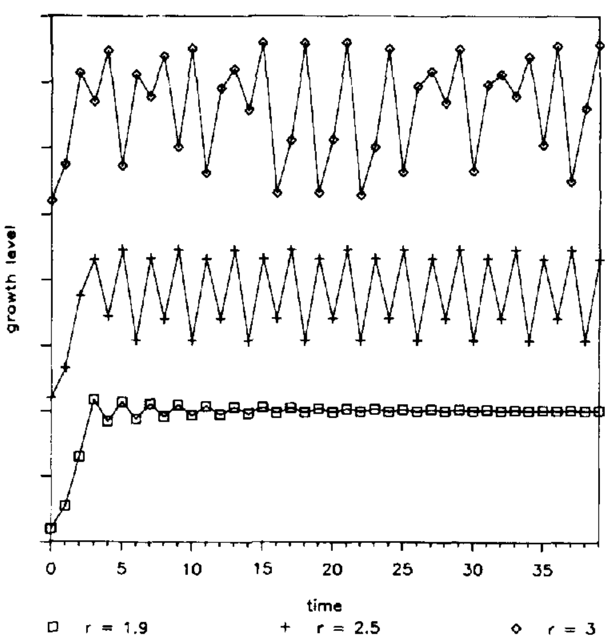

Figure 8. A smooth and linear increase in growth rate corresponds to abrupt shifts in the form of logistic growth curves (vibration toward stable level K for $r = 1.9$, four-period oscillation for $r = 2.5$, and chaotic oscillation for $r = 3$).

Inferring Intermediate Growth Points in the Difference Equation

The difference equation for logistic growth reduces a growth curve to a set of discrete points, at a distance of ($n \cdot f$) from the initial state. The inference of intermediary growth points is not trivial. For instance, if the logistic curve were to describe the growth of a fly population, the members of which die at the end of the season after having laid their eggs, there would be no intermediate points. If the equation describes vocabulary growth, it actually specifies a mathematical relationship between any pair of growth levels that are separated by a time interval f. That is, in this case, there are intermediary states, because it is unlikely that all words simply pop up together an interval f later. Rather, they will probably be released in an exponential way: The more easily learnable words emerge soon after the process of learning has started, and the closer the end of the f interval, the more words will emerge per unit time. Thus, given an initial state, a feedback delay time, and a growth rate, one may compute L_{t+f} and infer any number n of intermediary states by solving Equation 24a for r':

$$L_f = (1 + r')^n \cdot L_0. \quad (24a)$$

It is easier to solve Equation 24b, however, which is based on Equation 7:

$$r_e = [\ln(L_f) - \ln(L_0)]/n. \quad (24b)$$

This yields r_e, which is an exponential growth rate, for n, the number of intermediary steps wanted. Any intermediary step between L_0 and L_f may be taken as a point for applying the logistic growth equation. Thus, the growth curve may be filled with intermediary points to any desired level of detail.

Exponential release of items probably occurs only at growth intervals that are very close to the minimal structural growth level, that is, close to the real initial state of growth. A general algorithm to make growth interpolations for irregular growth far from the initial state is the fractal interpolation method (Grasman, 1990). It is based on the concept of self-similarity. If there is an irregular growth process, with quasi-random ups and downs, the sequence of intermediary points between any two given measurement points can be assumed to follow a pattern that is geometrically similar to the pattern of the overall process. I use this property in a reconstruction of intermediary points in the growth of lexical knowledge.

Robustness in the Face of Random Perturbation

In real life, cognitive growth and learning are probably subject to all sorts of random fluctuations, for instance fatigue, fluctuations in the condition of health, random fluctuations in the quality and quantity of the information available to a learner, and so forth.[6] The logistic growth model is quite robust in the face of these random fluctuations, and this is so for two reasons. The first reason concerns the nature of the variables involved. The way in which growth rate and carrying capacity are defined implicitly takes the normal, random fluctuations of life into account. Carrying capacity is defined as the maximal stable growth level attainable, given all the resources and limiting factors of a cognitive environment. This implies that normal, random fluctuations, insofar as they interfere with cognitive growth, have been taken care of in the form of a resource property contributing to setting a specific limitation on the height of the carrying capacity (comparable to the way in which forgetting is accounted for by the height of K). Because the carrying capacity and the growth rate form a sort of weighted sum of a great variety of variables, very significant random fluctuations are needed to cause small to moderate changes in either carrying capacity or growth rate.

The second reason that the logistic growth equation (the difference form) is rather robust in the face of random perturbation lies in the form of the equation itself. For instance, if at some (or even each) step of the computation of a growth sequence, a random number of about 10% is added to or subtracted from the growth rate and carrying capacity, the resulting growth curve is still very similar to one where no such significant random numbers have been added. Note, however, that if L has approached K very closely, the variance of L is greater or smaller than the random variance of K, dependent on the mean value of r (high average rs producing random variance in L that is much higher than the random variance in K) and the random variance of r.

Note also that robustness is just one side of the coin. The effect of random perturbations is state dependent. For instance, in some well-determined regions of growth processes, even very small random perturbations have important long-term effects, for instance because a random perturbation pushes the system over a threshold value or because in some regions small random perturbations are of about the same magnitude as the structural changes themselves. In general, however, it may be stated that ordinary simple logistic growth processes are quite insensitive to normal random perturbations or moderate random variations in the carrying capacity and growth rate.

A Measure for Growth Efficiency

The subsumption of cognitive growth under ecological principles, particularly that of the struggle for limited resources, implies that growth processes have a price. That is, the growth of vocabulary or the increase in mathematical skill requires time and energy. Provided the "price" of a growth process could be estimated, its efficiency might also be calculated, and by so doing the relative efficiencies of different parameter values (e.g., different growth rates) could be compared. It is intuitively clear, for instance, that a delayed growth process with a growth rate higher than 2.57 is considerably less efficient than one with a rate of 1. The first uses time and energy to run through a chaotic oscillation and builds up high growth levels that are torn down immediately afterward. The growth process with $r = 1$, on the other hand, evolves toward a steady-state level of 1 and stays there. One may assume that a final growth level that is about the height of the carrying capacity is most optimal from a cognitive economy point of view: It is stable and provides a reliable basis for the growth of additional skills, rule systems,

[6] Fatigue and motivational changes may be an effect of the growth process itself; for example, motivation may decrease as a function of effort invested in the growth process. In this particular case, such changes are not considered extrinsic "random" factors.

concepts, and so on so that no resources are left unused. A simple operationalization of growth efficiency, therefore, is the average relative distance between the carrying capacity and the growth level, measured over a fixed time interval and starting with a realistically low initial state (e.g., 1% of K). Previously, I defined the relative distance between L and K, that is $(K - L)/K$, as the unutilized capacity for growth U. Now I call the average U over a fixed time interval U_m. Because the central issue is cognitive growth, that is, the processes that lead to the attainment of a steady state, the short-term efficiency of growth processes is measured, meaning the efficiency over the time needed to approximate the steady state. The more efficient a grower, the less time it needs to approach the steady state, and therefore the smaller the average relative distance between K and L, or U_m. Because the "cost" of learning a word cannot be compared, for instance, with that of learning to solve a fraction problem, I confine the comparison of efficiencies to one grower at a time (i.e., I compare the efficiency of different growth rates in vocabulary, for instance, but I do not compare vocabulary with learning to solve fractions).

Given a set of growth processes that are identical except in one variable, which in general will be growth rate, there is at least one for which U_m is minimal: $U_m^{\min}$. I can arbitrarily set the cost in terms of time, energy, effort, and so on of $U_m^{\min}$ to 1. Thus, the relative growth cost of a grower A (e.g., a grower with a growth rate r_A) is

$$E_L = U_m^A / U_m^{\min}, \tag{25}$$

where E_L is a measure of the relative "expenses" made to let L increase.

Provided the carrying capacities for all the compared growers are similar, the only extra cost factor involved is the cost of maintaining different growth rates. It is assumed that the cost of maintaining a high growth rate is higher than that of a low growth rate. A high growth rate implies a higher speed of acquisition and thus requires better information handling and structuring. From an ecological point of view, the maintenance of a more complex structure is more expensive in terms of resources needed than that of a less complex structure (similar principles occur in thermodynamics; Atkins, 1984). The cost of maintaining the average growth rate that led to $U_m^{\min}$ may be arbitrarily set to 1, and the cost of all other growth rates may be expressed as follows:

$$E_r = r_m^A / r_m^{\min}. \tag{26}$$

Let w_r be a weight factor attached to E_r, which, if E_L and E_r are considered to contribute equally to the total growth costs, is set to 1. To compare growers with different carrying capacities, one should reckon with the fact that maintaining a specific carrying capacity level uses resources and thus contributes to the expenses. Consider the most optimal carrying capacity level to be set arbitrarily to 1—whatever "most optimal" may mean under specific circumstances—such that the cost involved in maintaining this carrying capacity is 1 too. Then attribute a weight w_K to it. The final efficiency equation is

$$E_G = (E_L + w_r \cdot E_r + w_K \cdot E_K)/(1 + w_r + w_K). \tag{27}$$

Figure 9 represents an efficiency graph for an ordinary logistic growth curve with delayed feedback, with an initial growth level of 0.1 and a time interval of 20 growth steps, which is sufficiently long for most growth rates to approach the final state level (if any exists). Given the chaotic nature of growth where $r > 2.57$, the right part of the graph is not very interesting. However, as can be seen from Figure 9, there are two local optima, namely about 1.5 and 1.86. These growth rates correspond to the approximate growth type explained in an earlier section: It is a very fast growth form characterized by a sequence of over- and undershootings approaching the K level. It is the sort of growth that can be expected in fast and early forms of learning, such as the learning of the meaning of words.

Although the differences between the optima and their neighboring points are small, such optima may be interesting from a cultural–evolutionary point of view (see Boyd & Richerson, 1985; Lumsden & Wilson, 1981). Assume that there exist two alternative forms of a skill that have largely similar sorts of functional meaning (e.g., ways of solving social conflicts either by democratic or by autocratic decision). Suppose further that these alternatives are currently evenly distributed over the population and that their learning by a new generation is largely a matter of imitation. If both strategies are equally difficult (or easy) to master, their distribution over the population will remain identical over consecutive generations. However, if one is easier to master—in the sense that its appropriation goes more efficiently than with the alternative—a slight increase may be expected in the number of people from the next generation who have adopted the more efficiently learnable strategy, and this relative increase will be proportional to its higher efficiency. This means that, all other conditions being equal, of any two alternative strategies, rules, or whatever whose germinal state is imitation, the most efficiently learnable form will become dominant in a population and finally exorcise its less efficiently learnable competitor. However, the effect of acquisition efficiency is sensible only under conditions where efficiency is a decisive property of a growth or learning process, that is, under conditions of extreme survival pressure. Such conditions have most probably occurred often during the early ages of the human species. This could explain why basic universal human properties such as language and social structure are learned very easily, at an early age, and in a more or less approximate way. Of course this is mere speculation, but it could be tested by mathematical theories of cultural evolution, such as Boyd and Richerson's.

Building Dynamic Systems Models of Cognitive and Language Growth

General Principles

The model of cognitive growth discussed so far may be extended considerably by applying principles of dynamic systems modeling. Basically, a dynamic system consists of a state space to which a set of dynamic rules is assigned. A state space is any n-dimensional space, the dimensions of which consist of the various degrees of freedom of a system. For instance, the growth of vocabulary during the one-word stage can be described in the form of a state space consisting of four dimensions, namely the number of words acquired, the growth rate, the height of the carrying capacity, and the feedback delay. For a real-word learner, the state space will consist of a fragment of a

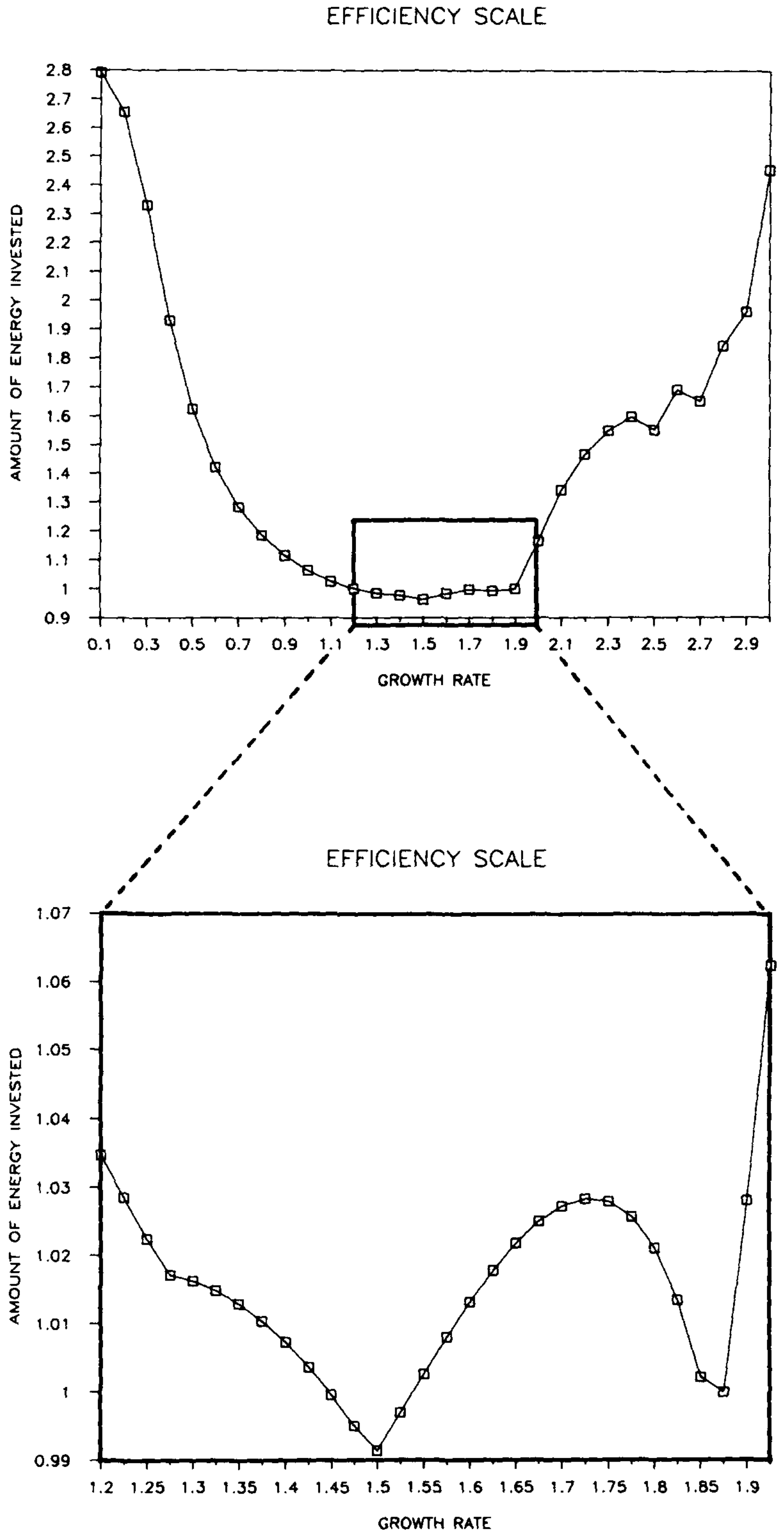

Figure 9. Efficiency graphs for different growth rates, where $0.1 < r < 3$; growth rates $1.2 < r < 1.9$ are roughly equally efficient (top); the window $1.2 < r < 1.95$ shows two local efficiency maxima, at $r = 1.5$ and $r = 1.85$.

Euclidean four-dimensional space, with number of words and carrying capacity between 1 and 500, growth rate between 0.5 and 2, and a feedback delay of between 1 and 2 weeks. Each point in this state space represents a potential developmental state of a word learner. The set of dynamic rules assigned to the state space describes the evolution over time of any point in the space. In the example, the dynamic rule consists of the difference form of the logistic growth equation. Thus, for a point (K= 1, L = 0.1, r = 1, and f = 1), the equation describes a line that consists of an arbitrary number of consecutive states. This particular starting point can be described as the *initial state of the system.*

The line starting from this initial state, which may be alternatively called the *trajectory of the initial state point,* amounts to the logistic growth form discussed in the previous section. The trajectory for initial states where $r < 1$ leads asymptotically toward a point where $L = K$. This point is an attractor for all trajectories where $r < 2$. For $1 < r < 2$, the trajectories move toward the point attractor in a whirling fashion. For $2 < r <$ 2.57, the attractor is no longer a point but a cycle (i.e., the periodic oscillation discussed previously). With linearly increasing r, these cycles get twisted to form twofold, fourfold, eightfold, and upward loops. For $r > 2.57$, the trajectories form extremely complicated open loops. Points for which r is 1, 2, and 2.57 are *bifurcation points,* or points where the geometry of the trajectories changes qualitatively (e.g., from a spiral into a closed loop). Because a four- and often a three-dimensional state space is difficult, if not impossible, to represent on paper, it is necessary to compress the dimensions some way. A good way to compress the dynamic system of logistic growth is to use two dimensions, namely L/K (i.e., growth relative to K) and a dimension representing absolute speed of growth. The latter can be expressed as $(L_{t+f} - L_t)/L_t$. Because for any given L_t this speed of growth is a direct function of r, this second dimension is a way of representing different values of r in the state space. Figure 10 shows the growth types of Figure 8 in the form of their state space diagrams. Such diagrams can reveal structure or regularity that would not be visible in the ordinary time-axis diagrams.

Instead of the geometric concept of a dynamic system, from now on a more intuitive version is used. By *dynamic system,* I intend any structure of n one-dimensional variables that affect one another over time. The way in which they do so is expressed in the form of difference equations for logistic growth with different parameters. Thus, the simplest dynamic growth system is the K–L–r–f system (or simply K–L–r system, because I treat f as a constant), where only L develops over time. In the discussion on the ecological constraints on cognitive growth, I explain that cognitive growth dimensions may entertain supportive or competitive relationships. For instance, vocabulary and syntactic knowledge are two growers that seem to compete for the same resources, whatever those may be, in the first stages of language development (Dromi, 1986). Each grower may be represented by a K–L–r structure, and both structures may be related by a mutual competitive relationship. For instance, one may postulate that each growth level inversely affects the growth rate of its competitor. I use the principle of competition to introduce a model of regressive or U-shaped growth. Another principle I mentioned earlier is that the growth equation may be applied recursively. For instance, the growth rate of vocabulary may be conceived of as a variable that is subject to growth in its turn and thus can be characterized by a growth rate r_r and a carrying capacity K_r, and so in principle ad infinitum. This principle of recursion, in addition to the principle of supportive growth, is used to introduce a model of bootstrapping phenomena in cognitive growth. Finally, dynamic models are constructed where the direction of growth (i.e., increase or decrease) is treated as an *autocausal* phenomenon; that is, increase causes increase, and decrease causes decrease. This form of positive feedback effect can be used in models of stepwise and U-shaped growth. In the remainder of the article, I present several empirical cognitive growth phenomena and try to construct a dynamic systems model for each of them. These models will consist of structures of K, L, and r components and their respective competitive and supportive relationships. Instead of studying the geometric properties of the state spaces corresponding to these structures, I discuss ways of fitting empirical growth curves with trajectories that naturally occur in these spaces. The study of several adjacent trajectories in these state spaces may reveal specific models of families of growth curves of which the empirical growth curves are members.

To provide a short overview of the types of bilateral interactions (i.e., interactions among any two variables in a system), I return to the five heuristic principles mentioned earlier, and the second principle in particular. This principle makes a distinction between competitive, supportive, and neutral interactions between variables. Thus, provided that the effect of one variable on another may be one of the aforementioned sorts, there are nine possible bilateral interactions (Table 1). The column heads specify the effect of a variable A on a variable B, whereas the row heads describe the opposite, that is, from B to A. The most interesting interaction types are 1, 2, and 4 because the neutral type is only an interaction in the logical sense of the word (it is a zero-order interaction). Type 1 is a mutual support interaction: Both growers, A and B, positively affect each other's growth. This type of interaction is used in models of bootstrap dynamics. Type 4 represents a mutually competitive interaction: Both growers negatively affect each other. This type of interaction will occur in competition among alternative strategies and in cognitive takeover phenomena. Type 2 represents an asymmetric form of interaction: Whereas one variable affects another positively, the latter affects the former negatively. This type of interaction has played a prominent role in biological models of predator–prey relationships and is usually modeled in the form of so-called Lotka-Volterra equations. Interactions of the Lotka-Volterra type can be used to model the growth of corrective behavior of parents, for instance, in response to the growth of unwanted habits of behaviors in their children (van Geert, in press).

Table 1
Supportive, Competitive, and Neutral Interactions Among Variables

Interaction	Supportive	Competitive	Neutral
Supportive	1	2	3
Competitive		4	5
Neutral			6

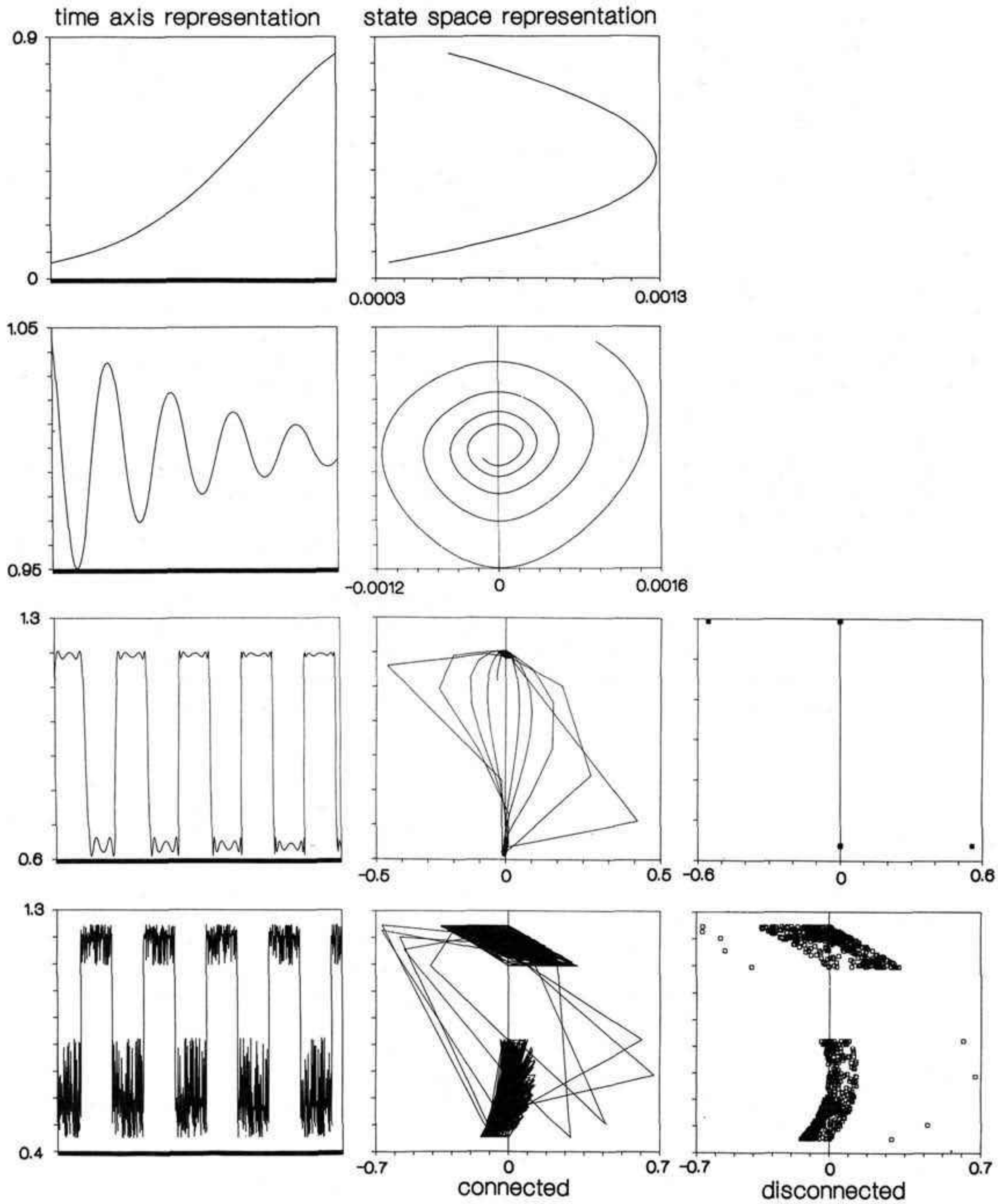

Figure 10. Growth curves can be represented in different ways: Figures at the left represent normal time-axis diagrams; figures at the right are corresponding state space diagrams. (State space diagrams exist in two forms: line representations, which show the connections between data points [connected state space diagrams], and point diagrams, which represent only the data points [disconnected state space diagrams]. Connected state space diagrams provide clear pictures of approximative and oscillatory growth [second and third rows]. The disconnected type of diagram is especially useful for representing chaos and near-chaos evolutions: Local concentrations of data points can be seen immediately [bottom row].)

Cognitive Dynamics Based on Mutual Competition

Cognitive takeover phenomena and competitive growth. Under various conditions, the growth in one dimension may force another dimension to change qualitatively. For instance, a multicellular organism that exceeds a specific number of cells is, through evolution, forced to abandon its direct cell-environment contacts as a major form of energy exchange and to develop an inner structure (e.g., a structure with a digestive tract). Cognitive and linguistic rules and strategies can be considered as information-management and production systems. Just as with the biological example, the form of information systems is not independent of quantitative properties of the information they have to manage. Manageability is a function of resources; that is, it poses no problems if one has all the time, space, and energy of the world. If an information-management system reaches a management limit—for example, in addressing problems that require more information than the system can cope with because of memory limitations—then it is likely either to remain constrained by that limit or to be replaced by another, more powerful information-management system. If the latter is the case, it is called a *takeover*. In the previous examples, the takeover is the effect of scarcity in internal resources, such as time or memory extension. In other cases, takeovers may be the result of external resource limitations. For instance, children often use "wrong" linguistic forms that are based on their immature grammars. Adults initially accept such forms but become increasingly intolerant as the child's mastery of the correct form increases.

Cognitive development is full of takeover phenomena. Examples are the takeover of one-word by two-word sentence rules, takeover of concrete operatory logic by formal operatory logic in specific domains of application, or the takeover of one balance scale rule by another in the balance scale task (Siegler, 1983). In a significant number of cases, takeover phenomena usually result in developmental regression (Bever, 1982), U-shaped behavioral growth, or oscillatory growth (Strauss, 1982; Strauss & Stavy, 1982). Such regression is either a complete abandoning of the older rule or principle or a temporary decrease of the field of an initially successful application of a specific production rule, concept, and so on. Regressions have been found in the fields of early object cognition, concept development, ratio comparison, early imitation, language acquisition, face recognition, artistic development, intuitive thinking, gender identity development, and so on (Bever, 1982; Strauss, 1982). A good example of regression toward zero performance level caused by a quantitative increase in the information to be managed by a rule system is provided by the development of conservation in 2- to 5-year-olds (Mehler, 1982). According to Mehler, 2-year-olds use a perceptual memory strategy and are capable of correctly solving a significant number of simple conservation problems. As perceptual differentiation increases, the amount of information encoded within a situation to be remembered increases correspondingly. At a specific point, the average amount to be remembered is simply too much to be managed by the perceptual memory strategy. This is the point at which a new strategy, based on the inference of rules and regularities, is adopted, while the old strategy quickly disappears. The new strategy, however, leads to a spectacular reduction—toward zero—of correct conservation performance. Only at the age of 5 is the regularities strategy back at the performance level of the 2-year-old using the memory strategy. The regularities approach, however, is much more powerful and has a much higher performance ceiling.

A comparable example concerns the development of the concepts of temperature and sweetness studied by Strauss and Stavy (1982). Children aged 4 use identity justifications and manage to solve about two thirds of the problems correctly. Then they start to use a method of comparing quantitative properties, which results in a considerable drop in the number of correct solutions. By the age of 11, they start to understand the connection between identity and comparison rules and solve most of the problems correctly. The main question, then, is why at a specific moment should one strategy (representation system, etc.) take over the domain of another that has proved to be quite successful? Why does the cognitive system not choose to adapt to the carrying capacity or resource limitations level and stick to its earlier and successful strategy? One explanation could be based on a Wernerian *orthogenetic principle:* The cognitive system would show an intrinsic tendency toward more complexity and rationality. That is, a logically or informationally more powerful system will inevitably interfere, at one moment or another, with a less powerful one. Without claiming that this solution is necessarily false, I present an alternative answer that is not based on a postulate of intrinsic tendency toward higher complexity but simply on a dynamic systems model of competitive growth.

The dynamic system consists of two growers, A and B. A is the strategy, rule, or whatever that, at the end of the day, will be the weaker of the two (e.g., the qualitative strategy in the sweetness or temperature problems or the memory strategy with conservation). This relationship between A and B can be expressed by stating that the final state of A is considerably lower than that of B, or that the carrying capacity of A is much smaller than that of B (e.g., at the end of the day, the rule strategy will solve many more conservation problems correctly than will the memory strategy). In principle, the competition between the two growers is mutual; both "suffer" from the success of their competitor. A simple way to introduce this competition mathematically is by making the growth parameter r of a grower A increase or decrease by some direct function of the difference between its growth limit, that is, carrying capacity K_A and the actual growth level of the competing grower B, namely L_B (and similarly for the growth rate of B; see Figure 11). The expression $(K_A - L_B)$ indicates how much of the potential application field of a grower A is already accounted for by a competitive grower and, if $L_B > K_A$, the powerfulness of the B form compared with the A form. It thus provides a good characterization of the relative success of A versus B (and vice versa). Because the growth parameter r is also supposed to be constrained by some intrinsic resource level (otherwise, it would quickly mount toward a rate that far exceeds the physical possibilities of its carrier system), the logistic growth equation can be applied to the growth parameter itself. For instance,

$$r_{a_{t+f}} = [1 + (K_{A_t} - L_{B_t}) \cdot c] \cdot r_{A_t} - (K_{A_t} - L_{B_t}) \cdot c \cdot r^2_{A_t} / K_{r_{A_t}} \quad (28a)$$

specifies that the value of r of the grower A at time t + f is a logistic function of several variables, namely (a) a previous value of r of A at time t, (b) a carrying capacity K for r_A at t, (c) a growth rate that amounts to the difference between the carrying capacity for A, K_A, and the growth level of the grower B, L_B at time t, and (d) a competition factor c (the bigger the value of c, the stronger the effect of the competitor).

The system consisting of two competitive growers A and B is characterized by four equations, namely

$$r_{A_{t+f}} = f(r_{A_t}, c_A \cdot (K_{A_t} - L_{B_t}), K_{r_{A_t}}), \quad (28b)$$

$$L_{A_{t+f}} = f(L_{A_t}, r_{A_t}, K_{A_t}), \quad (28c)$$

$$r_{B_{t+f}} = f[r_{B_t}, c_B \cdot (K_{B_t} - L_{A_t}), K_{r_{B_t}}], \text{ and} \quad (28d)$$

$$L_{B_{t+f}} = f(L_{B_t}, r_{B_t}, K_{B_t}), \quad (28e)$$

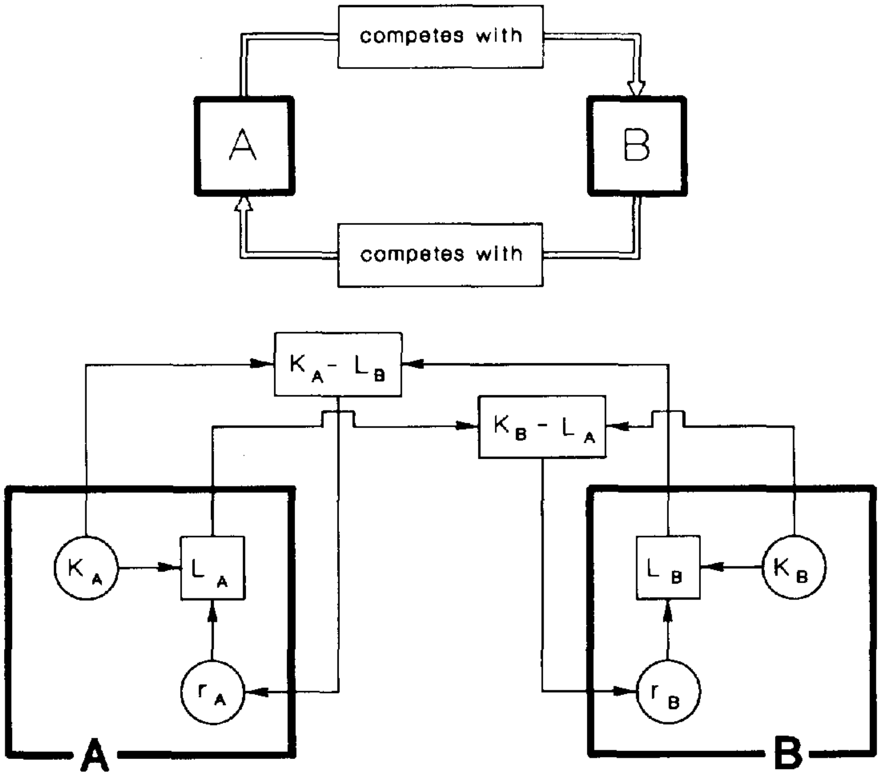

Figure 11. A dynamics model of two competing growers: The simplified model (top) specifies only the direction of the negative effect of one grower on another as a consequence of a competitive relationship. (In the present model, A competes with and thus negatively affects B and vice versa. The full model of interactions [bottom] specifies the way in which the competitive interaction has been shaped: The growth rate of A is a function of the difference between the carrying capacity of A and the growth level of B [and vice versa for B]. Thus, the bigger L_B, the smaller r_A.)

provided that

$$K_{B_i} \gg K_{A_i},\ L_{A_i} \gg L_{B_i},\ r_{A_i} \gg r_{B_i}, \qquad (28f)$$

where f is the symbol for the logistic function; the subscript i refers to the initial state values. This set of equations describes a system consisting of a more primitive strategy A, which is rather well developed at the start of the observations (hence, the initial state values of L_A and r_A are significantly higher than those of L_B and r_B, for example, the identity strategy in Strauss's sweetness experiment). Its growth initially profits from the lack of competition from a more powerful strategy, which is then still in a sort of germinal state (e.g., the quantitative rule in the sweetness experiment).

Assume that unrealistic values of the competition factors—that is, those that lead to highly instable and chaotic behavior of the dynamics—have been ruled out in the course of the phylogeny of the human cognitive system. Given realistic competition factors, the behavior of the dynamics modeled after Equations 28b–f is as follows. If the strong grower B (i.e., the grower that will achieve a much higher final level than A but with an initial state level that is significantly lower than that of A) has a reasonably high initial growth rate, both growers develop as if their growth was completely independent of the growth of their actual competitor. That is, in this particular region of the state space of competitive growth, there is no regression (fallback or U-shaped growth). For a significantly lower initial growth rate of B, grower A sets into a chaotic oscillation as B approaches and passes A's carrying capacity level. As B approaches its own carrying capacity level, the amplitude of A's oscillation decreases, and finally A settles down into its final state value (see Figure 12; see the Appendix for details). Probably there are various ways in which this sort of growth may be expressed in behavior. For instance, the oscillation of A amounts to quick alternations in the use of the A-versus-B strategy in problem solving. This is the sort of phenomenon that often occurs when a more mature cognitive strategy goes through some sort of breakthrough state (Bijstra, van Geert, & Jackson, 1989; Thelen, 1989). If one takes the average growth level in such a dynamics (e.g., over three consecutive states), one finds a temporary regression. However, the following sections discuss much more direct forms of the regression phenomenon.

Positive feedback effects in U-shaped growth. Cognitive strategies, rules, and knowledge are reality-driven acquisitions. If a cognitive strategy continuously leads to errors, it is likely to be abandoned; that is, it will show negative growth or extinction. Extinction or negative growth can be easily modeled in the logistic growth equation by inverting the sign of the growth rate from positive to negative. If such inversion were to happen whenever a strategy encounters an instance of counterevidence, cognitive growth would be very chaotic, if not simply nonexistent. However, according to cognitive developmental studies, as well as from studies in the history of science, children and

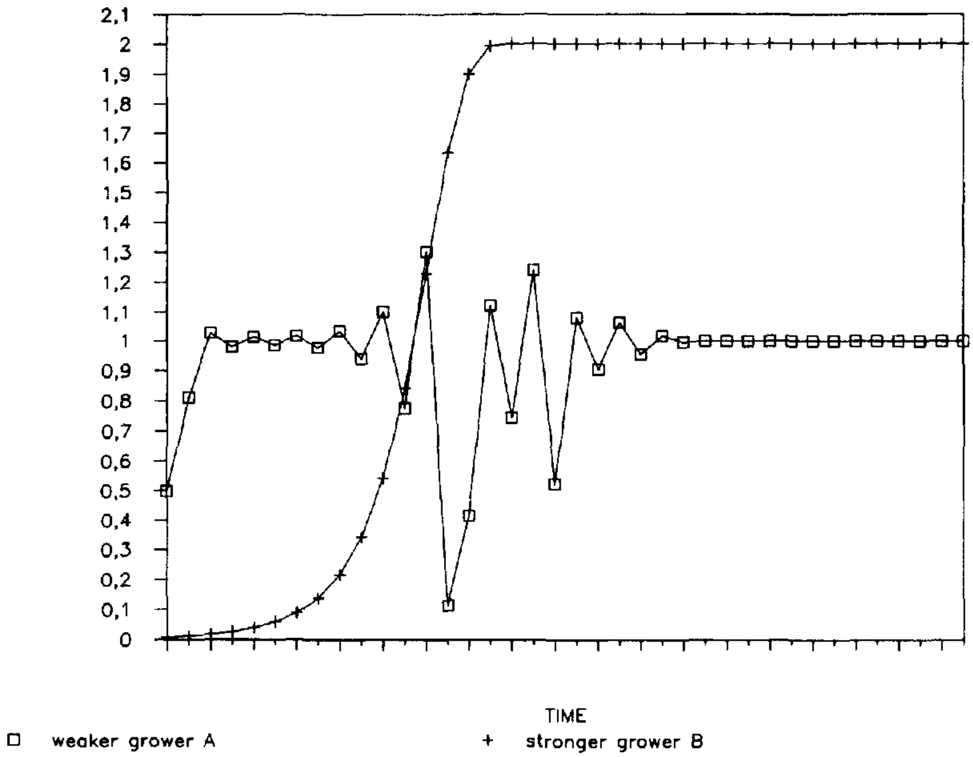

Figure 12. A cognitively weaker strategy *A* becomes temporarily unstable as a stronger strategy *B* takes over its domain of application, and *A* stabilizes again as *B* stabilizes.

scientists alike tend to stick to their models, theories, world views, or strategies, even in the face of considerable counterevidence (Elbers, 1986, 1988). That is, cognitive development is rather conservative. However, once a paradigm shift has started, it leads rather rapidly to considerable changes. A good way to build this behavior into the growth models is the following *positive feedback rule:* When increasing, continue increasing; when decreasing, continue decreasing. Thus, as long as a strategy or rule grows toward some carrying capacity level, it will continue doing so; that is, it will not invert the sign of its growth rate. However, if the growth rate is higher than 1, the grower will overshoot its carrying capacity and inevitably be thrown back under the carrying capacity level. This fallback then switches the sign of the growth rate (in accordance with the simple conservative principle just mentioned), and an extinction process sets in. This process will continue until the grower passes some minimal structural growth level. This level is the minimum level of a strategy, rule, principle, and so on, given the specific structural possibilities of the cognitive system at issue. For instance, an operationally thinking child will not completely abandon qualitative strategies in solving sweetness or temperature problems. To the cognitive system, such strategies are well within the structural possibilities and are thus likely to be generated anew even when, for some reason or another, they might have disappeared temporarily. Thus, as soon as a strategy in the process of extinction falls under its structural minimum, it automatically rises again, which switches the sign of the growth rate, and its growth starts anew (provided, of course, that neither *r* nor *K* has meanwhile been reduced to about zero level). This mechanism, which is based on a positive feedback effect of the direction of growth on the sign of the growth rate, provides a model for the so-called cognitive pendulum phenomenon (K. E. Nelson & Nelson, 1978) and also for regression in cognitive growth. The model accounts for the fact that strategies, rules, models, and so on are changed only if their application effectively exceeds the carrying capacity of the environment, that is, the level above which the strategies and other things always and automatically lead to errors or to anything else that is unpleasant to the system, such as a taxing of resources.

One may object that the conservative rule just mentioned makes growth extremely vulnerable to random perturbations. This is true only if the random perturbations are bigger than the increases or decreases due to either growth or extinction. If that is the case, the grower as such is intrinsically unstable anyhow, in that the growth level depends more on random factors than on cognitive acquisition or learning capacities. Nevertheless, small random perturbations have very interesting and far-reaching effects on growth, as I show later. First, however, I specify the dynamic rules for a system of competing growers to which the aforementioned positive feedback has been added and then demonstrate the effect of small random perturbations, which have very interesting long-term consequences on growth.

The equation for the growth rates in positive feedback systems is actually a modification of Equation 28a, namely

$$r_{A_{t+1}} = \text{abs}[(1 + (K_{A_t} - L_{B_t}) \cdot c) \times \text{abs}\, r_{A_t} - (K_{A_t} - L_{B_t}) \cdot c \cdot r^2_{A_t}/K_{r_{A_t}}] \cdot v_t \quad (29a)$$

for

$$v_t = (L_{t+f} - L_t)/\text{abs}(L_{t+f} - L_t), \tag{29b}$$

which implies that v_t is −1 if the last growth level is lower than its immediate predecessor and +1 otherwise. The complete set of dynamics rules is similar to Equations 28b–f except for the equation on which the growth rates are based (Equation 29). I demonstrate the behavior of the resulting dynamic system for initial state values $L_A = 0.5$, $r_A = 0.8$, $L_B = 0.1$, and a minimal structural growth level, which is almost 0. For varying initial state values of r_B, there is the following behavior. For the initial growth rate $r_B > 0.33$, A and B simply grow seemingly independently of one another; that is, no regression occurs. For r_B with a value of about 0.24, A shows inverted U-shaped growth, which, depending on the exact value of r_B, may amount either to complete or to partial extinction (see Figure 13; see the Appendix for details). For r_B with a value of about 0.1, there is a prototypical U-shaped growth, as described by Bever (1982), Mehler (1982), and Strauss and Stavy (1982; see Figure 13). For an r_B of about 0.07, grower A displays M-shaped growth, and for increasingly lower initial growth rates for B, the growth of A turns into a chaotic oscillation, exhibiting W-, UM-, or other-shaped growth (Figure 13).

If a small random perturbation p is added to each successive growth state of A and B (for instance, between −0.005 and 0.005), the growth patterns are no longer determined by the initial growth rate values. Overall, the same sort of growth patterns are found as with the deterministic system, but their connection with the initial values is only probabilistic. The fate of the competitive system is now strongly determined by those small random perturbations that occur in the vicinity of the carrying capacity level (e.g., 1) on the one hand and the minimal structural level (e.g., a very small positive number) on the other hand. This property is reminiscent of the so-called butterfly effect in some dynamic systems (the name refers to the fact that, in principle, the weather for the next day could be determined by a butterfly fluttering its wings at the proper time and place; Gleick, 1987; Schuster, 1988). The form of the growth curve A is explained by two factors. First, it depends on the course of the growth rate itself, the absolute magnitude of which is approximately equal to the first derivative of the growth level of B (see Equation 28b) and thus assumes a bell shape. Second, it depends on the local positive or negative value of the growth rate, and this depends partly on the magnitude of the random perturbation in the vicinity of the carrying capacity and structural minimum levels (see Figure 14).

The model of U-shaped growth presented thus far is only the most elementary model. First, it does not specify the way in which performance patterns are mapped upon the growth of the competitive cognitive strategies in the model. One mapping might consist of the subject following the strategy providing the highest success rate (e.g., as in U-shaped face recognition, Carey & Diamond, 1977). This results in a set of potential performance patterns that vary from growth patterns containing one plateau (of increasing length) to growth with real fallbacks, all dependent on the speed with which the weaker strategy goes up and down and the stronger strategy grows (see Figure 15; see the Appendix for detail. Another pattern might consist of a random oscillation between both available strategies as the older starts to decline (e.g., in conservation intermediates). Finally, if a strategy drops nearly to zero, this might force the child to apply a new strategy that is not yet sufficiently developed to deal successfully with the problems he or she now has to face, which inevitably leads to high error levels (Bever, 1982; Strauss & Stavy, 1982). Still another possibility is that the actual performance is itself a grower, the growth rate of which is a function of either the growth level or the growth speed of the strategies A and B. Which of these possibilities actually models empirical performance curves should be decided through a thorough empirical investigation of the growth process in question.

Second, the resource level or carrying capacity for the A grower has been held constant in the examples. It is likely, however, that this level also increases with age, for instance because of increases in supporting cognitive strategies or because of a bootstrapping effect (discussed later). Correspondingly, the structural minimum level of a grower might increase as an effect of general cognitive growth. This implies that the regressions will become shallower the later they occur in the course of the growth process.

Third, if individuals vary significantly as to relevant initial state properties, it is very likely that group data will actually conceal the U-shaped nature of the underlying developmental dynamics. An empirical description of a U-shaped growth process, and of any process of cognitive growth for that matter, should amount to a description of a state space, where the trajectories correspond to individual growth curves. The principle of positive feedback effects in cognitive growers will turn out to be a useful building block of more complex cognitive dynamics, for instance in dynamics explaining stepwise growth or irregular growth toward an optimum (discussed later).

Competitive growth in alternative strategies. The previous sections focus on the problem of takeover of an inferior by a superior strategy (regardless of how this superiority had been defined). However, there are many instances of cognitive growth where, in principle, no such value distinction exists among alternatives. An example is cognitive "styles" in strategies, such as globalistic versus analytic problem approaches, a more democratic versus a more authoritarian way of solving social decision problems in a peer group, and synthetic versus analytic perception of class distinctions. A criterion for distinguishing such alternatives from developmentally unequal strategies or skills is that the former will show a distribution over a population that is relatively independent of developmental levels (interindividual distribution) or that a single individual will use different alternatives in different contexts (intraindividual variation). A basic idea behind the growth of alternatives in a subject or in specific problem contexts is that the final dominance of one strategy over the other is the result of a competitive growth process among the alternatives. Such competition can be achieved by making the growth rate of each alternative linearly dependent on the growth level of its competitor. Thus, for two alternative strategies A and B, the growth equations are as follows:

$$L_{A_{t+f}} = (1 + r_A + c \cdot L_B) \cdot L_{A_t} - r_A \cdot L_{A_t}^2 / K_A \tag{30a}$$

$$L_{B_{t+f}} = (1 + r_B + c' \cdot L_A) \cdot L_{B_t} - r_B \cdot L_{B_t}^2 / K_B \tag{30b}$$

(the factors c and c' are negative numbers expressing the magnitude of the negative effect of B on A and vice versa).

Despite the simplicity of the equations, a system of two com-

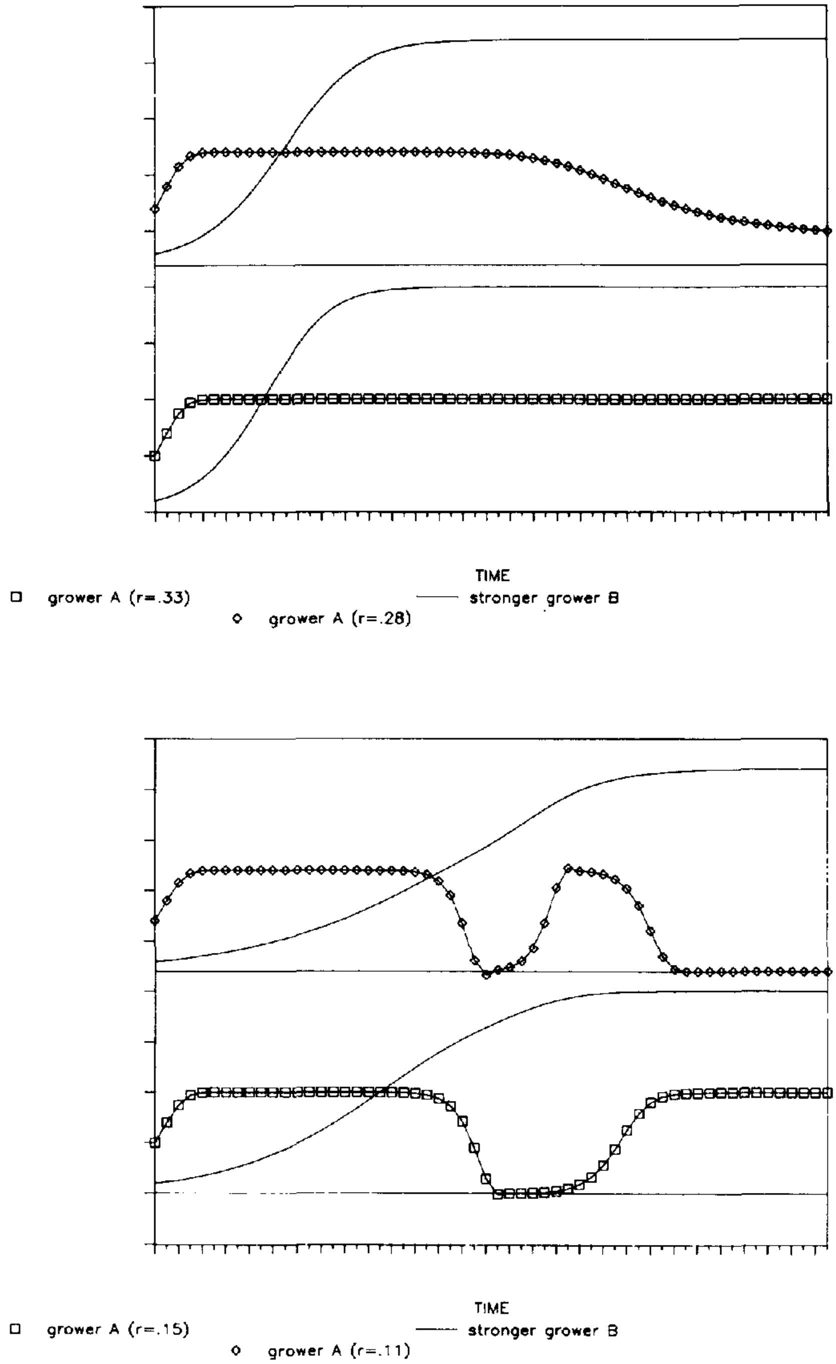

Figure 13. The growth patterns of competing growers, one of which is subject to positive feedback, are different for different initial growth rates of the stronger grower B; for $r_B > 0.33$, no regression in the weaker grower A occurs (grower A is indicated by a line formed with either squares or diamonds).

peting alternative strategies shows a variety of qualitatively distinct outcomes (see Figure 16). First, for growth rates exceeding a specific threshold level (about 0.1), the alternative strategies A and B grow toward a stable level, which, if the growth rates of A and B are not too different, is approximately equal to

$$L_A = K_A \cdot r_A/(r_A + K_A \cdot c)$$
$$L_B = K_B \cdot r_B/(r_B + K_B \cdot c') \qquad (30c)$$

(if K, r, and c are exactly similar for A and B, there is a single steady state that asymptotically approaches the value expressed in Equation 30c). In other words, if alternative strategies grow sufficiently quickly, they will evolve toward a steady-state ratio that approximates

$$[K_A \cdot r_A \cdot (r_B + K_B \cdot c')]/[(r_A + K_A \cdot c) \cdot K_B \cdot r_B]. \qquad (30d)$$

This ratio corresponds with the probability that a person will use either alternative A or B in a specific problem situation.

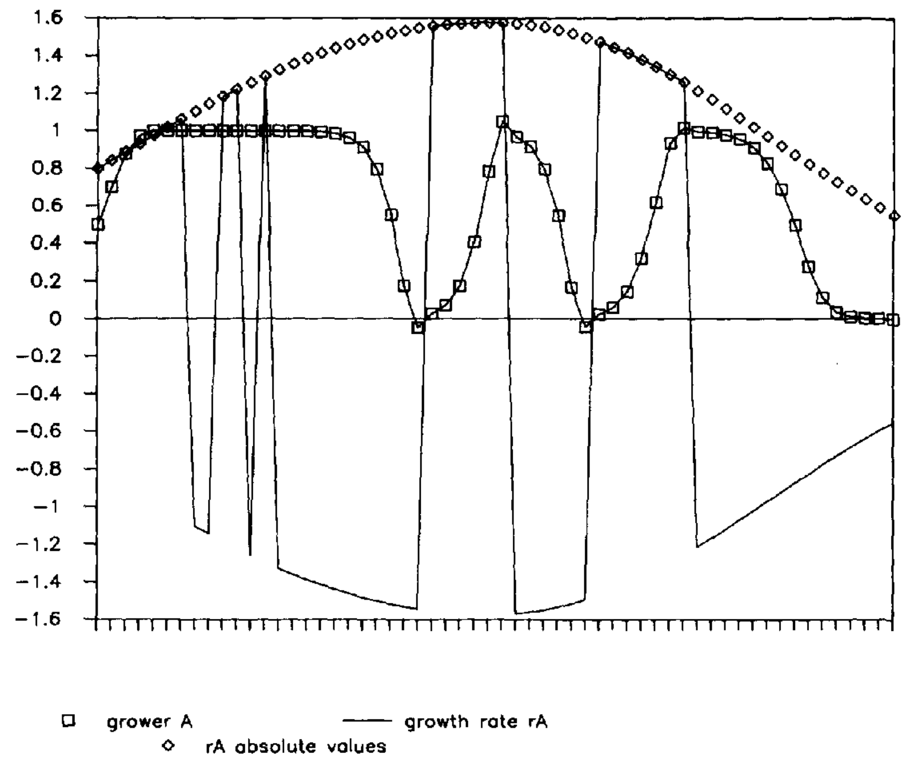

Figure 14. The growth rate of the weaker grower A changes over time. (While the absolute height of r_A follows a bell-shaped course, the actual direction of growth expressed by the sign of r_A changes around the extreme zones of 0 and 1; such changes are sensitive to small random perturbations only in the vicinity of Zones 0 and 1.)

Second, if the growth rates stay beneath a threshold value (e.g., if $r = 0.05$), a completely different type of behavior is found. Provided at least one of the parameters at the initial state is slightly unequal to its counterpart with the competing strategy, the most advantageous of the alternatives will grow toward its maximum, whereas its competitor, after an initial stage of increase, will drop back and evolve toward zero level (see Figure 16). The steady-state outcome (either A wins and B disappears or vice versa) depends on sharp threshold values that interfere in complex ways. For instance, a disadvantageous competition ratio (A suffers more from B than vice versa) can be compensated by a slightly higher initial state level, or a slightly higher initial state carrying capacity, or a slightly higher growth rate. The qualitative patterns that appear are the following: If one alternative is significantly more advantaged than the other, its growth resembles a normal logistic S shape; if both alternatives are less strongly different in regard to the major parameters, a stepwise growth pattern is observed in the winning alternative, and an inverted U shape is observed in the losing alternative. The winner qualitatively evolves from a zero stage (strategy absent) by means of an intermediary stage, where both alternatives have about equal growth levels toward a final stage characterized by complete absence of the competitor. This qualitative pattern is often observed in several forms of cognitive growth. In several cases, which depend on the exact ratios among the parameters, temporary regression may be observed in the finally winning strategy. The qualitative patterns that result from the present dynamic interaction between alternative strategies are preserved if parameters, and particularly the competition factor, vary randomly over time (e.g., under a ±10% variation of the default values). This qualitative conservatism of the dynamics is important in light of the fact that the parameters are unlikely to remain mathematically constant in

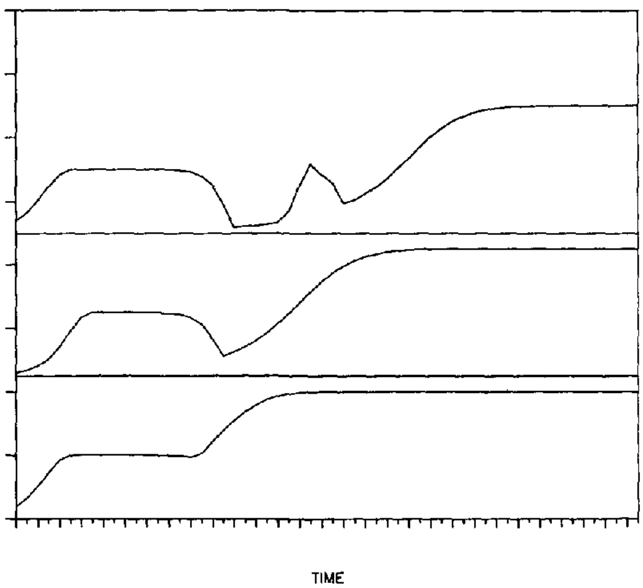

Figure 15. Potential performance curves based on competing growers, one of which is subject to positive feedback (curves follow the highest performance level).

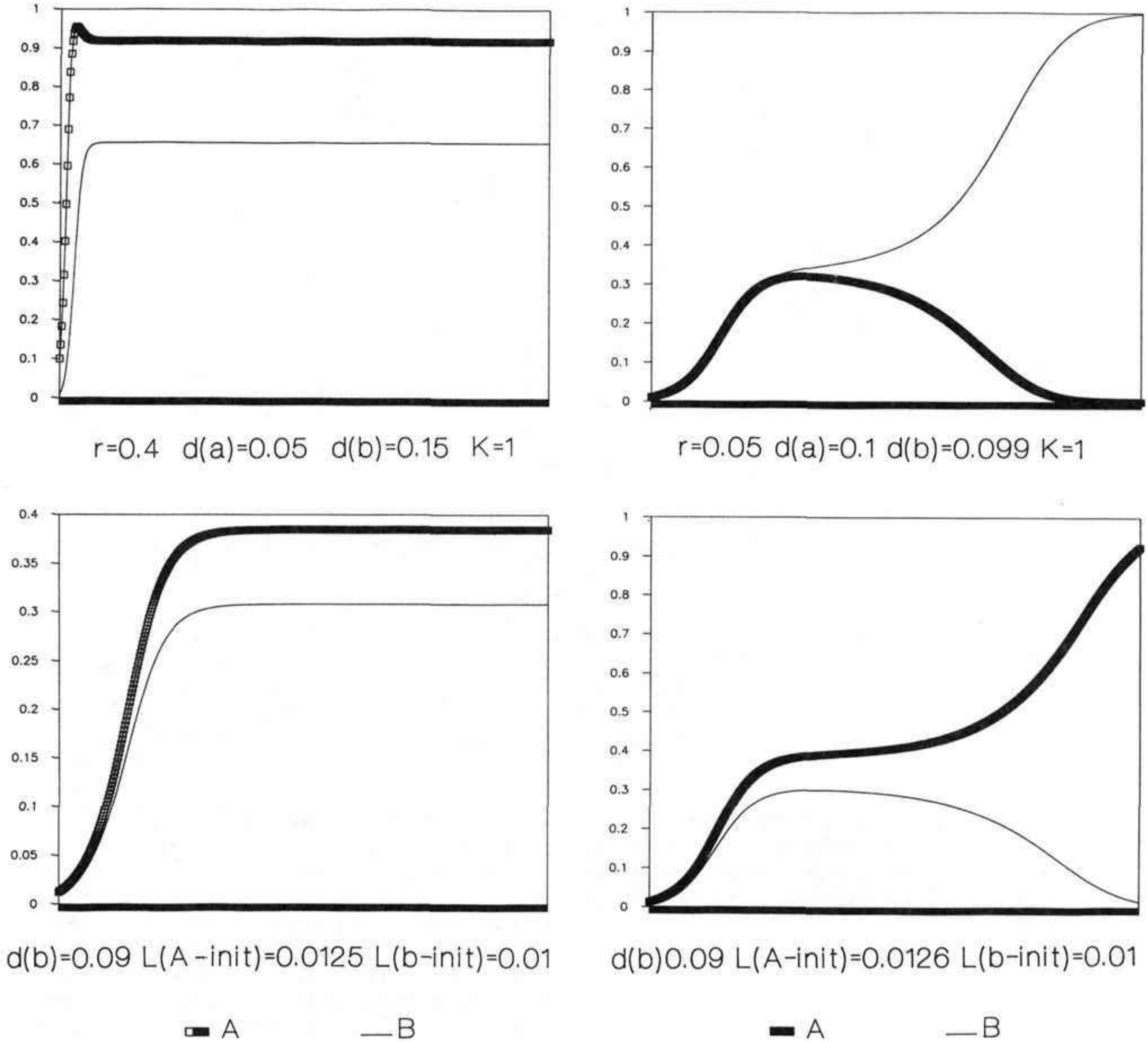

Figure 16. Threshold phenomena in competing growers: For a sufficiently large growth rate ($r = 0.4$) the competitors grow toward a stable ratio (top left). (With small growth rates [$r = 0.05$], very small inequalities in the competition factor lead to a disappearance of one and an upper limit growth of the other. *A*, which suffers slightly more from *B* than vice versa grows toward a maximum, then disappears. *B* grows toward a level equal to *A*'s maximum, then starts to increase as *A* decreases [top right]. If *A* compensates its competitive disadvantage with a slightly higher initial [init] growth level, *A* and *B* grow toward a stable ratio [bottom left]. A very slight increase in *A*'s initial state advantage changes the pattern into the opposite of the second case: *B* disappears, and *A* grows toward an upper limit.)

real cases. The question of what factors determine the value of the parameters cannot be answered without taking the context of the growth process into account. For instance, a holistic strategy will suffer less from an analytic alternative if the child in question is led to use this sort of strategy in other problem domains, whereas there may be more difficulties in competing with the analytic alternative if the latter is favored by the environment. Finally, the occurrence of threshold values and the compensatory interactions among the parameters are reminiscent of phenomena that may be observed in education in general and compensatory education in particular: the observation that success in changing unfavorable learning conditions depends on crossing threshold levels, the existence of which is often difficult to predict, and the fact that conditions of unfavorable learning can be changed via different ways and under specific conditions.

Cognitive Dynamics Based on Mutual Support

Bootstrapping: A basic cognitive growth mechanism. According to most theories, cognitive development amounts to a bootstrapping process: Cognitive growth releases the resources upon which further growth is based. For instance, in Piaget's theory, the child's activity is based on his or her current level of cognitive development. These activities bring about experiences that affect the underlying cognitive structures and lead these structures toward increasing complexity and equilibration (Piaget, 1975; van Geert, 1987a). In transactional models (Sameroff, 1975), the nature and level of the child's current development, for example, of temperament, is considered to release supportive activities in the caretaker and to change the norms, expectations, and supportive activities of the caretaker with regard to the child. In language development, the child's caretakers adapt the syntactic complexity of the language addressed to the child to the child's assumed level of understanding, and they speak so-called Motherese (Snow & Ferguson, 1977).

Bootstrapping dynamics are easy to construct. To the elementary *K–L–r* dynamics is simply added a supportive relationship from L to K (see Figure 17). That is, the initial carrying capacity for a low growth level, of some syntactic rule for example, is low also (and it is made so low because of some tutorial or pedagogical adaptation, e.g., because the parents temporarily tolerate grammatical errors and are not inclined to actively correct the child). The carrying capacity grows as an effect of increases in the growth level it supports. One way of structuring this relationship between L and K is to make the growth rate of K a function of the growth level L. In principle, there are two

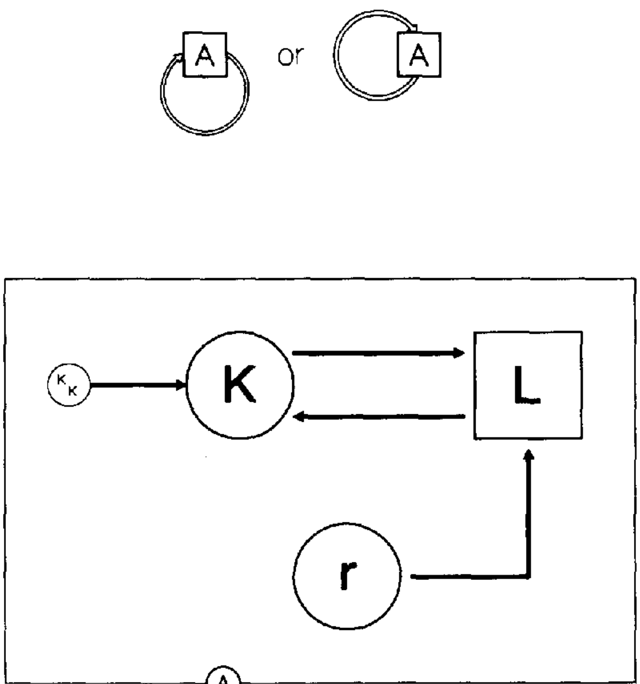

Figure 17. A model of a simple bootstrap dynamics. (The simplified model [top] represents a grower A positively affecting itself [circular arrow]. The full model [bottom] specifies this bootstrap effect: Arrows represent effects as described in the logistic growth equation. The arrow from L to K implies that L contributes to the growth of its own carrying capacity K. K is further affected by its proper carrying capacity K_K, which specifies an upper limit for K.)

ways in which the carrying capacity K can be a function of the growth level L, namely as a function of relative change in L on the one hand and the absolute level of L on the other. I shall first discuss a dynamics based on relative increase.

K's growth might depend on the relative increase in L, that is, on how much L has increased or decreased over the past period relative to its absolute magnitude or to its previous increase. This is the sort of increase that might be expected in transactional social models of development. For instance, in the interaction games studied by Bruner (1975; Bruner & Sherwood, 1976) and Wertsch (1979), a subtle interplay occurs between the information and guidance provided by the mothers and the behavior of their young children. The activity of the mothers technically amounts to a continuous raising of the carrying capacity of a growth variable (e.g., knowledge of language or of the structure of games) by raising the demands as well as the complexity of the examples and corrections given. One may assume, however, that mothers, or educators in general for that matter, will be sensitive to the relative growth of knowledge in the child, that is, to the speed with which the child progresses on the aspect of knowledge at issue. More precisely, the rate of making more complex help and information available probably will increase if the child proceeds quickly and probably will decrease if the child no longer shows considerable relative progress (this sort of feedback principle is nicely illustrated in Bruner, 1975). In addition, one may assume that mothers or caretakers in general will differ in the level of sensitivity with which they will regard the child's behavior as a signal of his or her needs and developmental level (Ainsworth, Blehar, Waters, & Wall, 1978). This sensitivity can be expressed in the form of a damping function, or a function that either magnifies or reduces the effect of a growth level on the growth of the carrying capacity:[7]

$$r_{K_{t+f}} = d \cdot (L_{t+f} - L_t)/L_t. \quad (31)$$

The dynamic rules for the complete system amount to the following:

$$K_{ef} = f(K_{ef}, d_K \cdot \Delta L, K_K) \quad (32a)$$

$$L = f(L, r, K_{ef}) \quad (32b)$$

$$d_K = f'(s_L) \quad (32c)$$

$$K_i \ll K_K, \quad (32d)$$

which reads for Equation 32a, the effective carrying capacity K_{ef} is a logistic growth function of its previous level, the change in the growth level L, a factor d_K damping the effect of L, and a maximal carrying capacity K_K; for Equation 32b, growth level is a logistic growth function of the effective carrying capacity K_{ef} and the growth rate r; for Equation 32c, the factor damping the effect of L on K is a function f' (e.g., a simple linear function) of the sensitivity of the system to changes in L, or s_L; and for Equation 32d, the initial level of the effective carrying capacity K_i is significantly lower than its highest possible final state level K_K.

[7] When no subscripts referring to time (e.g., subscripts t, t + f) are shown, it is assumed that the argument left of the equal sign should take the subscript t + f, whereas the arguments right of the equal sign take the subscript t.

Thus far, I have assumed that bootstrapping occurs in an upward direction. However, downward movements also occur in cognitive growth. For instance, it is likely that the initial growth rate of a fast grower like vocabulary is much higher than later growth rates, regardless of the damping effect of the carrying capacity (see the section on exponential growth without feedback delay). That is, especially with young children acquiring basic skills and knowledge, one may assume that the initial phase is characterized by very fast learning, whereas the learning rate will decrease as a consequence of increasing skill or knowledge level. Note that in the ordinary logistic model it is the absolute speed of growth that decreases as L approaches K, while the growth parameter r remains constant. The present model postulates that r itself shows a decline. Therefore, assume that the growth rate, of vocabulary growth for example, is also subject to negative growth, in that its carrying capacity, that is, the growth rate's potential final state, lies well below its initial state level. The bootstrapping principle may now be applied to the growth of the growth rate by assuming that the initial growth rate grows toward its much lower final carrying capacity as a result of the increase in the cognitive growth level (e.g., vocabulary). Thus, as far as changes in the growth rate are concerned, the bootstrapping leads downward, instead of upward, as with the carrying capacity discussed previously. For the sake of the argument, let me demonstrate the effect of absolute growth level (e.g., vocabulary) on the changes in the dependent variable (growth rate of the vocabulary). This effect amounts to the principle that the more words the child knows, the lower the rate with which the vocabulary grows, regardless of the actual carrying capacity for the vocabulary. That is, a competitive relationship is postulated from a growth level to its underlying growth rate. Again, one should assign some sort of damping factor to the variable vocabulary level to account for different levels of sensitivity within the dynamic system at issue. A dynamics consisting of a K–L–r structure with the decrease in r dependent on the increase of L gives rather trivial results (trivial in the sense that it yields growth curves where the growth rate decreases faster toward the end than with constant r). However, the competitive relationship from growth level to growth rate should be added to the dynamics discussed in the previous section, a dynamics that contained a supportive relationship from growth level to carrying capacity (see Figure 18). Thus, to the dynamics rules (Equation 32a–d) are added:

$$r_{ef} = f(r_{ef}, d_r \cdot L, K_r) \quad (32e)$$

$$d_r = f'(s_r) \quad (32f)$$

$$r_i \gg K_r, \quad (32g)$$

which reads for Equation 32e, the effective growth rate r_{ef} is a logistic growth function of its previous level, the growth level L, a factor d_r damping the effect of L on r, and a carrying capacity K_r, which amounts to a stable final level of r; for Equation 32f, the damping factor d_r is a function f' (e.g., a simple linear function) of the sensitivity of the growth rate r to the growth level L; and for Equation 32g, the initial level of the effective growth rate is significantly higher than its final state, which is close to K_r. I now give two empirical examples of bootstrap dynamics in which both forms of adaptations, upward adaptation of the carrying capacity and downward adaptation of the growth rate, are incorporated. In the first example, the main assumption is that parents adapt their help, correction, and support in function of the relative increase in a child's competence, that is, in terms of whether more or less progress has been made than during a comparable preceding period.

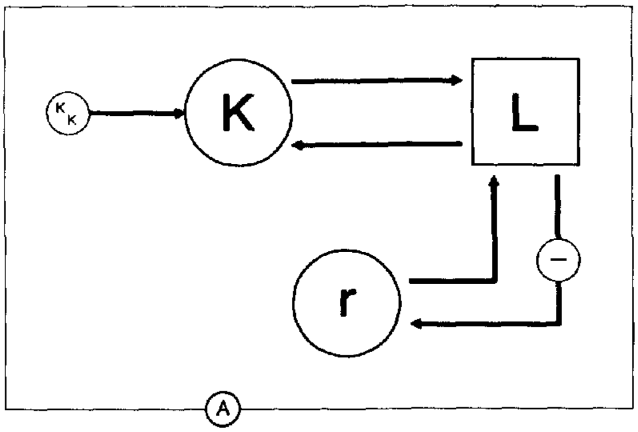

Figure 18. A model of a more complex bootstrap dynamics based on the model from Figure 17. (The arrow containing a minus sign specifies a relationship as described in the logistic growth equation, but with opposite sign. That is, L is a factor reducing, instead of increasing, the value of r. This bootstrap dynamics starts with a low initial K and a high initial r and stabilizes at a level where K is high ($K = K_K$) and r is low.)

A first empirical example: A bootstrap dynamics incorporating parental sensitivity. Ainsworth et al. (1978) have studied different forms of parental responsivity and their effects on the development of attachment in infants. Parents have been found to differ in their sensitivity to the infants' behavioral signals. This sensitivity is expressed in the form of responses toward these signals, in providing help and support, in giving the child a specific amount of independence, and so on. Mothers could be distinguished on the basis of differences in sensitive responsiveness: Some were highly (i.e., optimally) sensitive, others were highly insensitive, and a third group was characterized as inconsistently insensitive. The last group is characterized by a tendency toward overstimulating their infants (Belsky, Rovine, & Taylor, 1984), in addition to inconsistency in giving support (Ainsworth et al., 1978). Paternal sensitivity has an effect on the quality of the child's attachment (Belsky et al., 1984) and probably also on later personality and cognitive development (Campos, Barrett, Lamb, Goldsmith, & Stenberg, 1983; Van Ijzendoorn, 1988).

The dynamics modeling the developmental effect of parental sensitivity has the form depicted in Figure 18 and is modeled according to Equation 32a–g. In this model, parental sensitivity amounts to the damping factor that mediates the effect of the growth of a developmental variable (e.g., attachment behavior and sociocognitive understanding in the child) on the growth in the support and help given by the parent. This support and help should adapt to the increasing competence of the child. That is, it should take the form of a carrying capacity that increases as the growth level of the child's competence at issue increases. In the model, this increase in the carrying capacity is the effect of a growth rate, which is a function of the relative increase in a growth level of the child's competence and a damping function representing the parent's sensitivity. I do not specify the child's growth level L, but one might imagine it representing the quality of the child's attachment, for instance (assuming the latter is a continuous one-dimensional variable), a sociocognitive competence, the growth of which is believed to depend on the quality of the parent–child interaction, or a personality property such as ego control or ego resiliency (Block & Block, 1980). If the dynamics are run with a low-sensitivity factor, the carrying capacity, which is a function of parental support and responsiveness, grows slowly. In fact, it cannot keep a sufficient distance from the growth level of the child's competence L. As L comes within a critical distance of K, K and consequently, also L no longer increase; that is, they get trapped into a point of stability that is well beneath the maximal level that K and L could actually achieve (see Figure 19, top right; see the Appendix for details). With an optimal sensitivity factor, K and L grow smoothly toward their maximal level (maximal given the complete system of resources and possibilities in a particular environment; Figure 19, top left).

If the sensitivity is above optimum, the carrying capacity grows too fast, which corresponds to overstimulating the child (providing him or her with much more support and help than actually needed, given the child's present level of growth). The fate of high growth rates is that the resulting growth level starts to oscillate in an almost chaotic way. This also occurs in the present dynamics, and this corresponds with the observation that the parents' responsiveness is actually inconsistent (sometimes too high, sometimes too low; see Figure 19, bottom right). With increasing magnification of the sensitivity factor, the behavior of the carrying capacity (e.g., of the support and scaffolding of the parents) becomes chaotic, until it (almost neurotically) oscillates between its lowest and highest possible values. This usually results in a cognitive growth process that shows a mild oscillation around some low final state value. It is interesting to see, though, that the growth rate of the grower L may sometimes compensate for the damaging effect of oversensitivity. It may have a sort of appeasing effect on the neuroticism of the carrying capacity (e.g., the parents' deployment of resources, attention, and help) and lead toward a final state comparable with that achieved with an optimal sensitivity level (Figure 19, bottom left). One may therefore conclude that the present dynamics simulates the theory and empirical findings on parental sensitivity quite well, as far as the relation between the level of sensitivity, the resulting parental support, and the resulting growth in the child are concerned. From the previous simulation, it follows that the effect of decreasing sensitivity on the stable final state value of the growth level,[8] given all other initial state values are equal, is nonlinear (see Figure 20; see the Appendix for details). Running the dynamics with various sensitivity values demonstrates this clearly. As can be seen in Figure 20, the separation between optimal and hypersensitivity effects does not take the form of a simple dip in the final state effects. Rather, there is a chaotic band in which the final state effect of sensitivity values, which are very close to one another, is strongly different (i.e., either about zero or maximum). This chaotic zone separates two qualitatively different ways in which the environmental support system adapts to the needs of the grower (i.e., either with too high to hypersensitivity, or with optimal to low sensitivity).

A second empirical example: A bootstrap dynamics for syntactic growth. By *syntactic growth* I mean the increase in the relative amount of correct use of some specific syntactic rule. Examples are the growth in the correct use of plurals and present progressive, studied by Brown (1973), and the growth of inversion in *Wh-* questions (e.g., "What mama is doing" vs. "What is mama doing?"; Labov & Labov, 1978). Data on the growth of inversion are shown in Figure 21. To construct a dynamics model for growth of inversion, I make three assumptions. First, syntactic growth, such as inversion, is an example of a bootstrap process. That is, I assume that the carrying capacity, exemplified for instance by parental effort invested in providing examples of correct sentences and in correcting errors and by the child's effort in experimenting with particular syntactic constructions and paying attention to the latter, is a function of the growth of competence in using the syntactic rule already attained by the child. Second, I assume that in syntactic growth it is the absolute level of competence of the child to which parents are sensitive in regard to providing help, correction, and support. That is, growth in carrying capacity is not a function of relative increase in growing competence, but of the absolute level attained. In the present example, this absolute level has a psychologically relevant meaning, in that it can be compared easily with an absolute standard, namely correct grammatical use. Mature language users have very clear intuitions about the idiomatic grammaticality of sentences; ungrammaticality in the language of immature or nonnative speakers is quite salient information. This situation is different from the previous example (e.g., play behavior and event knowledge), where no such absolute standard exists or can be applied easily to characterize the child's absolute level of competence and where only relative increase provides useful information. Third, I assume that the empirical growth curve represents the growth in one underlying variable, namely the child's competence in using the inversion rule. This level of competence determines the probability that a correct form will be used (and inversely, that an error will be made; Spada & Kluwe, 1980). It is expressed in the amount of errors produced, relative to the amount of correct sentences.

The dynamics is of the type represented in Figure 18. The dynamics rules consist of rules presented in Equations 32b–g, but Equation 32a should be replaced by

$$K_{ef} = f(K_{ef}, d_K \cdot L, K_K), \qquad (33)$$

which reads that the effective carrying capacity K_{ef} is a logistic growth function of its previous level, the growth level L, a damping factor d_K, and a maximal final state level K_K. In accordance with the previous dynamics model, it is assumed that K increases toward a carrying capacity that yields 100% correct responses and that r decreases toward some minimum level. Instead of using constant damping functions, a curvilinear relationship may be introduced. That is, the effect of L on K and r is more strongly damped the closer L is to either its initial or its

[8] *Final state* has been operationally defined as the state after 100 iterative applications of the growth equation, that is, 100 f after the initial state); after this interval, all but the slowest trajectories have settled into a stable state, or at least into a state that changes only very minimally; if f = 1 week, 100 f is about 2 years, which is a sufficiently long developmental interval in view of the fact that the developmental processes discussed in this section take place during infancy and toddlerhood.

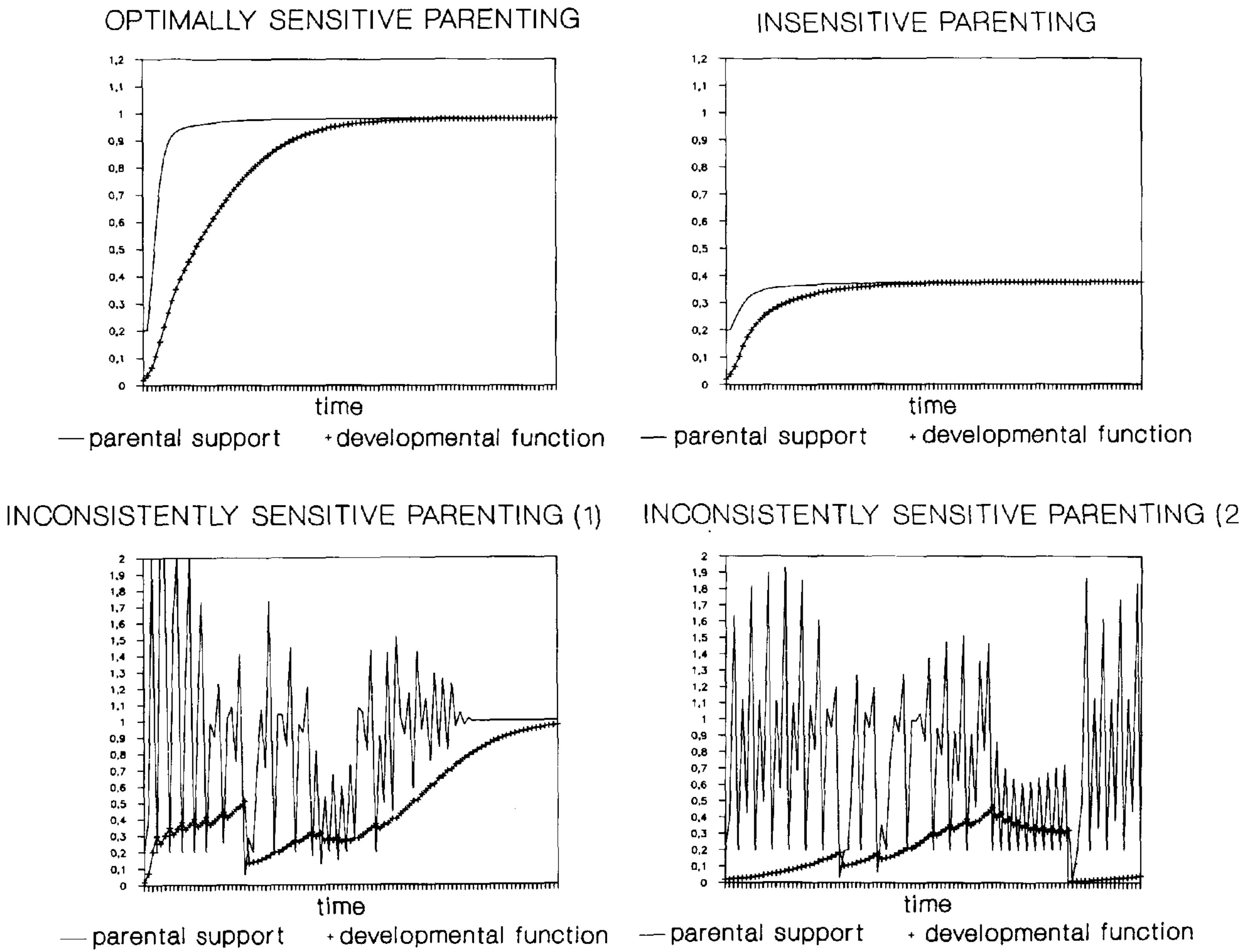

Figure 19. Parental responsivity and support determines the carrying capacity for a growing competence in the child; different sensitivity levels varying from hyper- to insensitive produce different growth patterns and final states. (The dynamics is described in Figure 18. The effect from L to K has been mediated by a damping factor, which is a linear function of the parent's sensitivity.)

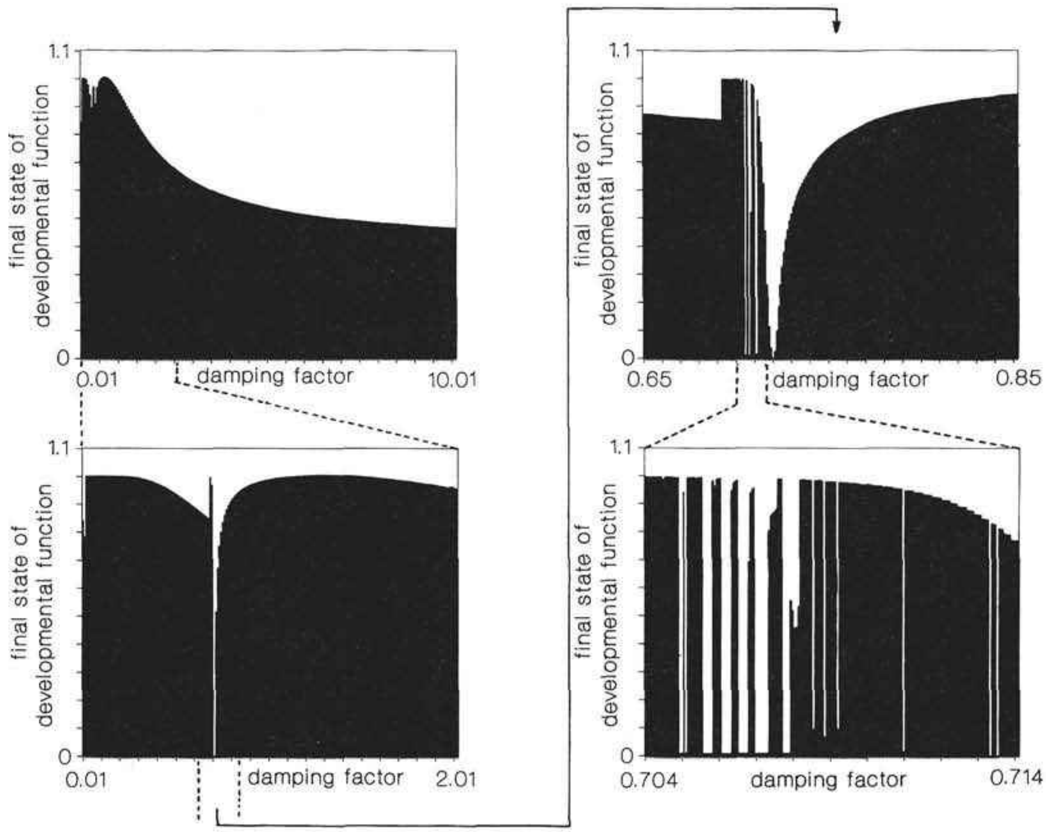

Figure 20. The effect of varying parental sensitivity levels upon the final state of a growing competence that depends on parental support, modeling, and so on. (Parental sensitivity is specified in the form of a damping factor. The figures represent increasingly small portions of the sensitivity scale, i.e., 0.01 < damping factor < 0.01, 0.01 < damping factor < 2.01, 0.65 < damping factor < 0.85, and 0.704 < damping factor < 0.714. In the latter region, the final state effect is strongly nonlinear, showing a chaotic succession of maximal and near-minimal final states. The chaotic nature of the dynamics in this region amounts to the fact that very small differences in the sensitivity parameter lead to fundamentally different outcomes.)

final state. The latter implies that the sensitivity of the system to the syntactic growth level is greater at the beginning than at the end (and the other way round for the former case). Figure 22 shows a simulation of the growth of inversion in *What* sentences (see the Appendix for details). Basically, it applies the principle of oscillatory or near-chaotic growth with a carrying capacity that itself grows as a consequence of the growth in the use of inversion in *What* sentences. The high initial growth rate grows toward a much lower final state, also as a consequence of growth in the inversion rule, thus accounting for a considerable reduction in the amplitude of the growth oscillations toward the end state (the 100% correct level). By intrapolating intermediary growth states[9] between the first and second points of the curve, a complete set of intermediary growth points can be computed (e.g., weekly data points, instead of monthly averages presented in the empirical study). Figure 22 shows two different interpolation strategies, one based on random numbers, another on the self-similarity method described earlier. In this case, the self-similarity method yields a strongly oscillating pattern, which is probably not in accordance with the empirical findings. It is also easy to study the effect of small random perturbations (e.g., ranging between −1% and +1% of the growth rate involved) on the evolution of the growth curve. It seems that, although the actual form of the curve may vary rather drastically as a consequence of such perturbations, the qualitative form of all these curves (irregular oscillations with diminishing amplitude as L approaches K) is rather robust. In fact, this finding is in accordance with individual data, which show rather strong intraindividual differences for different sentence types that are not strongly different in terms of complexity (Brown, 1973; Labov & Labov, 1978). Notwithstanding these intraindividual differences, the qualitative nature of these empirical curves is similar for all the types of syntactic structures studied.

Combining support and competition: An alternative explanation for syntactic growth phenomena. There appears to be an obvious fallacy in the previous dynamics model of syntactic growth: Correct use cannot grow higher than 100%, whereas the simulated growth curve may easily overshoot the 100% level,

[9] Instead of making an intrapolation on the basis of an exponential increase between the first and second growth points, it has been assumed that intermediary growth points follow an oscillating course geometrically similar to the overall oscillation of the empirical growth curve.

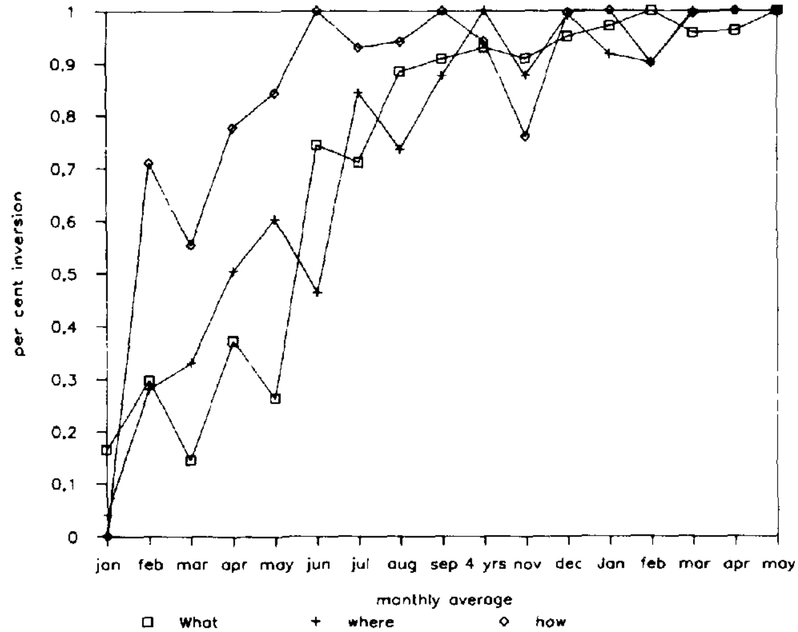

Figure 21. Empirical growth curves of percentage correct use of inversion in *wh-* questions. (Adapted from "Learning the Syntax of Questions," p. 23, by W. Labov and T. Labov, 1978, in R. N. Campbell and P. T. Smith, *Recent Advances in the Psychology of Language: Language Development and Mother–Child Interaction*, London: Plenum Press. Copyright 1978 by Plenum Press. Adapted by permission.)

provided growth rate is high. This problem could be solved by interpreting overshoot as error, as an overgeneralization error for example. More important, however, the model is rather unlikely from a psychological point of view. One of the major discoveries of cognitive developmental research is that the child's errors and mistakes are in general not based on trial and error, or sheer ignorance, but are rather the expression of underlying rules. This is true for cognitive and language growth alike. For instance, the use of the noninverse strategy in questions seems directly based on the subject–verb–object strategy (Quigley & King, 1980; Slobin & Bever, 1982), or on a "head-initial" parameter setting (Atkinson, 1986), or whatever rule is considered a major sentence-formation rule in the early stages of syntactic growth. Inversion, on the other hand, may be based on imitation of model sentences, an alternative setting of a syntactic parameter, and so on. Essential to this discussion is the understanding that these rules are competitive: The noninversion rule is consistent with major sentence-formation principles existing in the child's current grammar, whereas the inversion rule is consistent with the environmental sentence models. Assume that these rules are in fact separate growers, that their domains of application grow in a logistic way. Also assume that as the growth level of the correct rule (i.e., inversion rule) increases, the explicit support for this rule also increases, for instance because the child tends to notice more and more examples of this inversion rule in the language of the environment or because the inversion gets established as a structurally coherent rule in the child's grammar. Put differently, the carrying capacity for the correct rule increases as a consequence of a bootstrapping process. On the other hand, it is likely that the support for the wrong rule decreases as a result of growth in the use of the correct rule. For instance, parents probably tend to be more tolerant with regard to syntactic errors when the child starts to use a specific construction, such as a question, and less tolerant of errors as the child's capacity to use the correct rule increases; parents probably tend to provide less corrective modeling when a new sentence form has just emerged; and the child probably pays much less attention to parental corrections of syntactic constructions when a construction is new than later on. In fact, a whole range of factors contribute to the increase in the carrying capacity of the correct rule and the decrease in the carrying capacity of the wrong rule.

There could be an asymmetric competitive relationship between the growth level of the correct strategy and the carrying capacity (i.e., environmental support) for the wrong strategy. That is, the support for the correct strategy is probably directly dependent on the growth of that strategy, whereas the support (carrying capacity) for the wrong strategy decreases as the growth level of the correct strategy increases. Psychologically, the latter amounts to a decrease in the environmental tolerance of and support for the wrong strategy as the mastery of the correct strategy increases, whereas on the other hand, the support for the correct strategy does not decrease when, temporarily, the mastery of the wrong strategy increases. A wrong strategy is probably also characterized by the fact that it is tolerated until it exceeds some threshold (i.e., until it exceeds its environmental tolerance level, which is nothing other than its carrying capacity) and is then explicitly discouraged and rectified until its use falls below some minimal threshold (e.g., until it is no longer noticeable). After that, the wrong strategy may grow again until it again shoots above its tolerance, and the cycle may start anew. I have already described a dynamics for this sort of process, namely the positive feedback cycle in regressive growth (described in a former section), which is typical

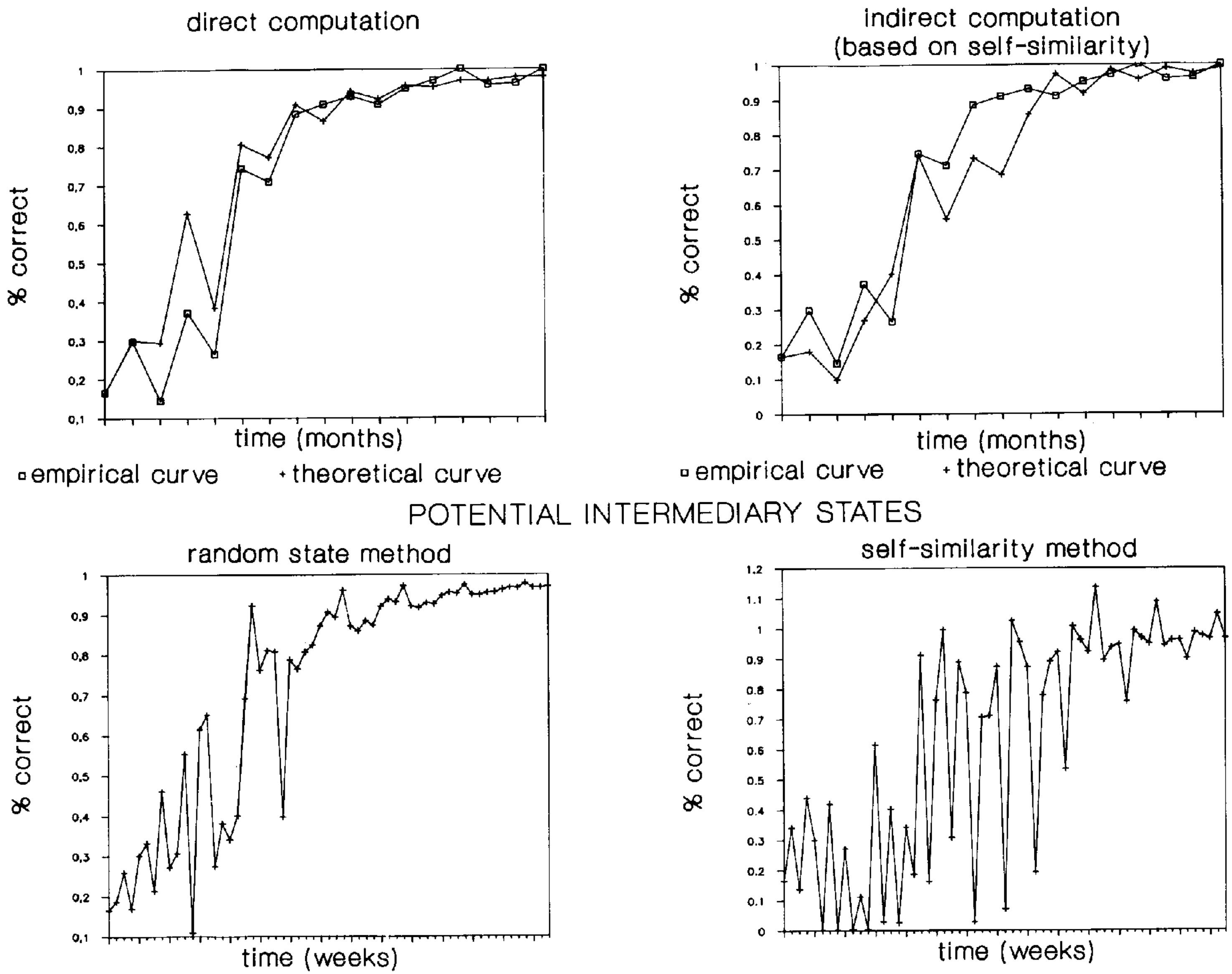

Figure 22. A mathematical simulation of growth of correct inversion rule in questions, based on a bootstrap dynamics (see Figure 18; top left). (Simulations of possible growth averages per week are based on two different methods: [a] inference of initial states during the first month based on random numbers with relatively small variance allowed [bottom left] and [b] self-similarity. The weekly data during the first 4 weeks are proportional to the data during the first 4 months. The latter method shows considerably higher instability than the former [bottom right]. By computing monthly averages over data reconstructed by the self-similarity method [top right], the validity of the underlying assumptions can be tested: The resulting curve deviates more strongly from the data than the curve based on a direct computation of the data points.)

of weak cognitive strategies. The complete dynamic system for the competitive growth of a correct and a false strategy is represented in Figure 23.

The dynamics rules for each grower are similar to those described under bootstrap dynamics (dependent on absolute growth levels) and regressive growth. Figure 24 (see the Appendix for details) shows a mathematical simulation of Brown's (1973) data on the present progressive in 1 child, which are based on setting initial state values of all the parameters involved in the dynamics and letting the system run in accordance with the equations described in the previous section. The curve shows the monthly average of the percentage of correct uses of the rule. Advantages of the dynamics are that it also yields a theoretical reconstruction of weekly averages and that it shows the growth of the correct and the false strategy separately (Figure 24, bottom). Such theoretically reconstructed data can then be checked against the empirical data as a further test of the postulated dynamic model.

Bootstrap dynamics of interacting and competitive growers are very rich, in that many different types of building blocks can be used (e.g., direct or indirect effects of L on K and r, effect of either absolute level of L or relative increase on K and r, and either positive feedback effects or not). Although each of these dynamics yields growth curves that are qualitatively very similar, the quantitative nature of these curves may be characteristic of different dynamics architectures. Much work still needs to be done to reveal the properties of these dynamics and to see how far they provide valid models of empirical cognitive growth processes.

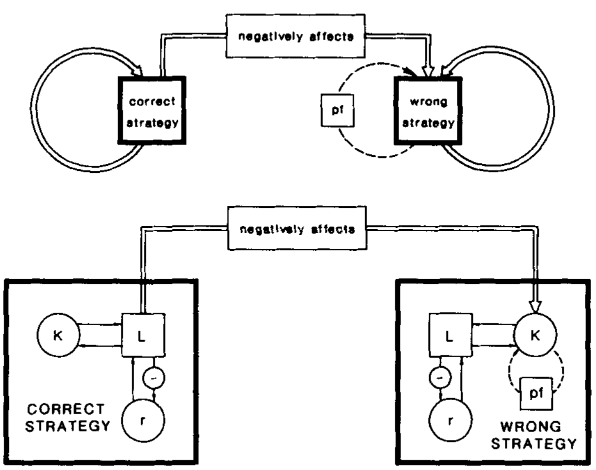

Figure 23. A dynamics for competitive growth in a correct and a wrong strategy. (The simplified model [top] specifies that both are bootstrap growers [the circular arrows refer to a positive effect of each grower on itself]. The correct strategy negatively affects the growth in the wrong strategy; i.e., there exists a competitive relationship from the correct to the wrong strategy. The wrong strategy is subject to positive feedback [the broken "pf" arrow]. The full dynamics model [bottom] specifies the nature of the negative effect of correct strategy on wrong strategy: The growth level of the correct strategy negatively affects the carrying capacity of [i.e., the support given to] the wrong strategy. The positive feedback cycle holds for the carrying capacity of the wrong strategy. Both strategies grow in accordance with a bootstrap dynamics as depicted in Figure 18 [with the K_K component omitted].)

Adaptation of the Carrying Capacity

In preceding sections, I discussed a particular form of adaptation of the carrying capacity called *bootstrapping:* The tutorial environment adapts its support to a current, low growth level and raises that support as a consequence of increase in the grower. This process amounts to a temporary adaptation of the effective carrying capacity, which then moves toward an intrinsic carrying capacity level that remained constant during the whole growth process.

I have shown that carrying capacity is a measure of the overall support a cognitive environment may lend to a specific grower. This specific amount of support is expressed in the form of a potentially stable upper limit a grower may attain, that is, the carrying capacity level. It follows that if major changes in the cognitive system occur, the inferred carrying capacity, for vocabulary growth for example, will change accordingly. The problem that is addressed in this section is the following: Because a grower (e.g., vocabulary) is an intrinsic part of the overall cognitive system and because the carrying capacity is determined by the overall properties of that system, what will be the contribution of growth in a single variable (e.g., vocabulary) to changes in the overall system and thus to changes in its own carrying capacity? In the context of this question, the bootstrap dynamics discussed in previous sections is a very particular tutorial adaptation of the carrying capacity. The carrying capacity as such is not changed, but it is dissociated into an effective carrying capacity and a sort of background carrying capacity toward which the effective K evolves as a consequence of growth in the dependent variable (e.g., grammatical rule use). This type of positive adaptation of support can also be observed in what Fischer and coworkers (e.g., Fischer & Canfield, 1986; Fischer & Pipp, 1984) called "practice and support" conditions of testing skills. Practice and support are offered in function of the student's increase in mastery of a skill, and this greatly enhances the speed with which the skill grows. A related concept is Vygotsky's "zone of proximal development" (Vygotsky, 1978).

An opposite form of adaptation occurs when, as a consequence of significantly low performance, for example in mathematics, a student's curriculum is changed (e.g., the student moves to another curriculum where mathematics is no longer an obligatory subject). These changes are coercive tutorial adaptations of the carrying capacity, and they are not the major concern of this section. More subtle changes in the carrying capacity might result from the negative effect that poor mathematics performance, for instance, might have on the student's attention to mathematics-related information, to effort spent in doing exercises, and so on. In this case, a slow downward growth of the carrying capacity for mathematics occurs as a consequence of low mathematics performance in the student. However, if a person's cognitive environment too easily lowered the carrying capacity of relatively slow growers (e.g., mathematics knowledge) and similarly raised that of quick ones (e.g., knowledge of pop music), the person concerned would rapidly change into some sort of "idiot savant" (it may be that that is what is characteristic of idiot savants, namely that their cognitive system adapts too swiftly to differences in growth rates of the components). I explore which principles of K adaptation are more adequate.

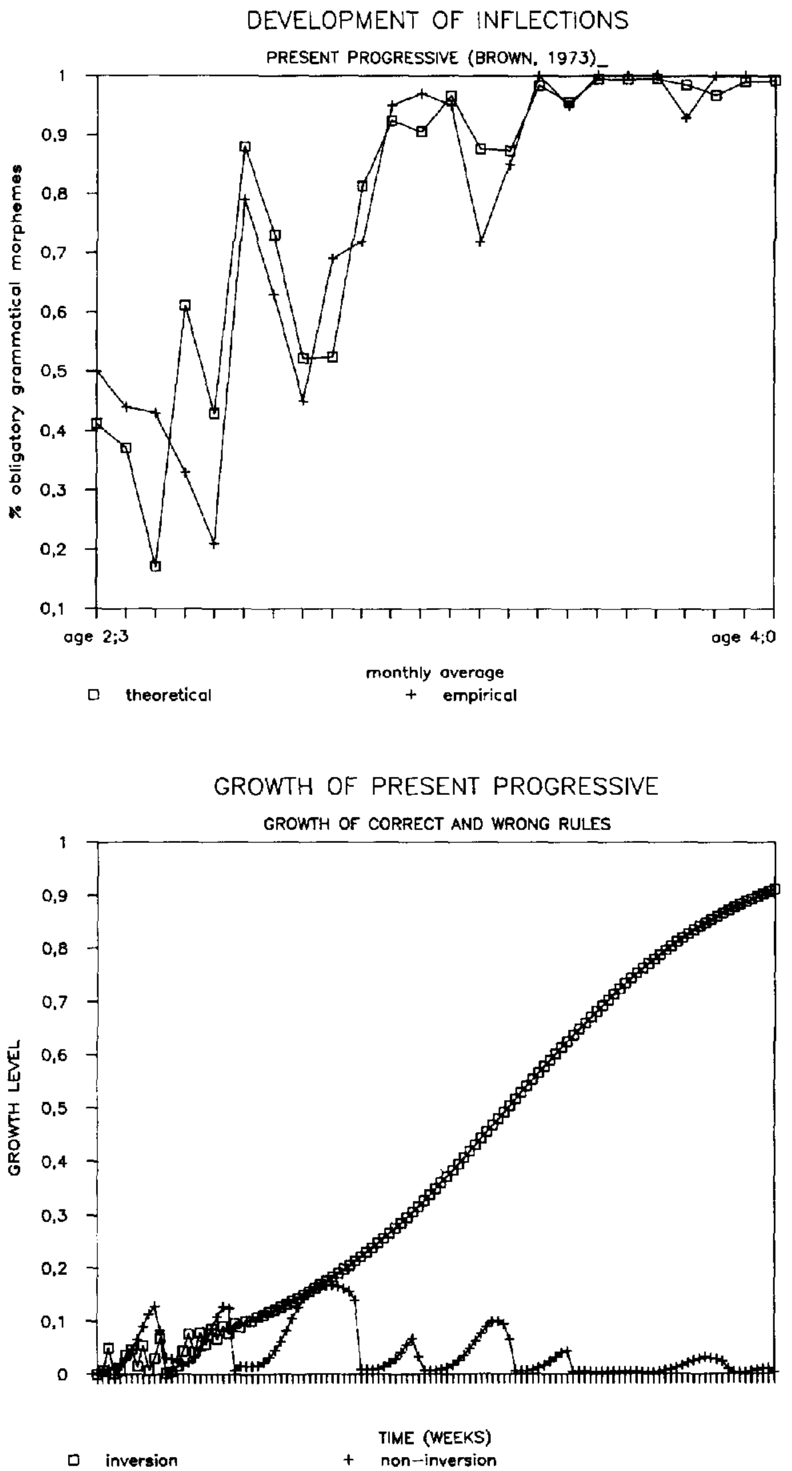

Figure 24. A mathematical simulation of the growth of the correct present progressive form; curve shows monthly averages of the percentage correct per week based on separate growth of correct and wrong strategy. (Although the curves [top] are slightly out of phase at the beginning, the theoretical curve follows the oscillations in the empirical data quite closely. The theoretical curve is based on a simulation of the growth of the correct and the wrong strategies [bottom]. The top panel was adapted from *A First Language: The Early Stages,* p. 256, by R. Brown, 1973, London: Allen & Unwin. Copyright 1973 by Allen & Unwin. Adapted by permission.)

A first adaptive principle implies that carrying capacities should grow slowly toward the growth level of slow growers, whereas a second holds that they should grow quickly away from the growth level of fast ones. Although the first principle is probably intuitively plausible, the second requires some explanation. Fast growers (e.g., $r > 1.5$) overshoot their carrying capacity level. The more they overshoot, the lower they will fall at the next step. It follows, then, that if K grows not toward but away from L, the next K level will always anticipate the next L level. This can be implemented easily by making K grow as a function of $(K - L)/K$.[10] A third adaptive principle is that K should grow as a function of the relative distance between K and L, that is, $(K - L)/K$, which is the unutilized capacity for growth, U. The default principle is that the growth rate of K should be higher as the distance from K to L is larger. These K-adaptation principles can be summarized in the following equations:

$$K_{t+f} = (1 + d \cdot U_t) \cdot K_t \tag{34a}$$

$$d = f'(r), \tag{34b}$$

meaning that the next state of the carrying capacity K is an exponential function of the previous state K_t and a growth rate, which is the product of a damping factor d and the unutilized capacity for growth U_t; d is a function of the growth rate r and yields a large negative number if r is very low, a number slightly bigger than 1 if r is very high, or a positive number that is practically zero for all intermediary values of r. The fact that an exponential instead of a logistic growth form is used for K_{t+f} implies that no intrinsic limits were set to the upper level that the cognitive grower may attain. It is expected that the upper limit of the grower will result from the way in which the growth rate r is determined by the parameters in the model. Figures 25 and 26 (see the Appendix for details) show the effect of K adaptation on slow and fast growers. The effect on a fast grower is of special interest: K increases and decreases such that oscillations will be damped, and K and L move toward a common stability point that is considerably higher than the original carrying capacity level (Figure 26). The natural interpretation of this effect is that people who are considerably better in some cognitive ability (mathematics, language, etc.) will achieve a higher level of mastery and, similarly, that cognitive environments tend to invest more resources into fast growers than in others. Using an efficiency equation (Equation 27), I can show that raising K and damping the oscillations is more efficient than keeping K constant and tolerating high-amplitude oscillations of the growth level. Such raising and damping is not necessarily the effect of intentional tutorial activities, but rather the result of a simple K-adaptation principle.

Figure 25 shows the effect of an adaptation of $(-U)$ to a slow grower, resulting in a considerable decrease of the final state of K. The fact, however, that the growth of K toward a slowly growing L depends on U, that is, on the relative distance between L and U, could be a disadvantage. Assume, for instance, that the growth of vocabulary in a child is very slow. A fast negative adaptation of the resources needed to build up a vocabulary (i.e., the vocabulary's carrying capacity) would finally result in a very poor vocabulary. Because in a complex cultural environment even a "minimal" vocabulary should be rather extensive, it is better not to adapt the carrying capacity too soon. One way to accomplish this is to increase the damping factor d from Equation 34a. Another way is to make K dependent not on $(-)U$ but on the inverse of $(-)U$: $(-)1/U$.

$$K_{t+f} = (1 + d/U_t) \cdot K_t \tag{35a}$$

$$d < 10^{-6}. \tag{35b}$$

Equation 35a implies that K adapts only very little as long as its distance from L is still big and adapts faster the closer K and L approach each other. The functionality of this form of adaptation lies in the fact that in the middle of the S-curve, absolute increase is rather considerable, even for low growth rates, whereas in the vicinity of K, the growth of L decelerates anyway. Thus, if K would wait to adapt to L until L has sufficiently closely approached K, implying that L's absolute growth would have decreased considerably, the system could have spared the cost of maintaining a high K when L is approaching only slowly, while still achieving a sufficiently high final state of L. Thus, U can be substituted in Equation 34a by $1/U$. The effect on growth of adapting K in accordance with this last equation is very interesting. Instead of settling down to a steady state, the grower and its carrying capacity start to meander in a sort of narrowband random walk, rather reminiscent of stock exchange variations (see Figure 27; see the Appendix for details). Dependent on the height of the damping factor, sudden leaps and dips may be observed. The whole system is also very sensitive to small differences in initial state conditions. For all practical purposes, the growth level and its carrying capacity behave as if they follow a random evolution, which in general stays within a small margin and now and then shows unexpected leaps and dips that are rather considerable. However, the evolution is not the result of random factors but is completely deterministic. More precisely, a deterministically evolving grower may provide a source of randomlike perturbations to other growers that depend on it. Thus, a growth system in which the present adaptation principles hold produces its own random perturbations. In complex systems, random perturbations are important in that they may determine the long-term evolution of the entire system, given that they occur at points where the system is in relative instability (Prigogine & Stengers, 1982; see, for instance, the section on pendulumlike growth processes and the effect of small random factors therein). Finally, in ongoing research, I am trying to determine the evolution of carrying capacities in terms of competition and support among growers in more complex systems (whereas in this article K has been treated as a single factor). It can be shown that systems of competitive and supportive effects from a multitude of growers on one another nevertheless result in one-dimensional K fac-

[10] For instance, when L is bigger than K, it will drop back under the K level, and the higher L is, the deeper it falls. If $L > K$, then the growth rate of K will be negative, and thus K will decrease relative to the distance between K and L. That is, at the next growth state, K and L will thus be in each others' proximity. However, the closer L is to K, the closer it will stay to K in the next growth state. Consequently, K and L evolve toward a stability point, which would not be the case if K would not adapt (see Figure 26).

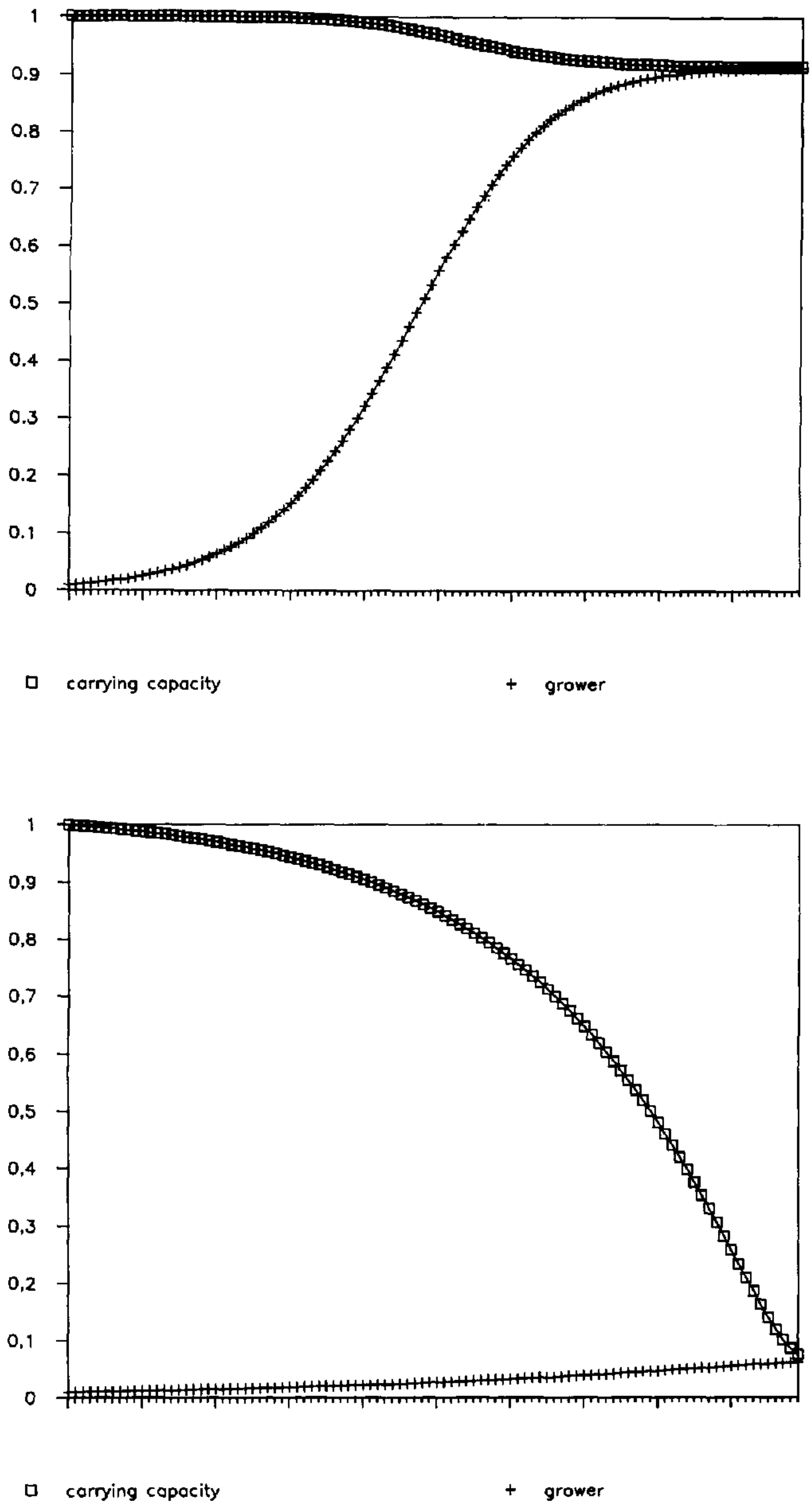

Figure 25. Adaptation of the carrying capacity K to a slow grower as a function of negative value of the unutilized opportunity for growth, $-U$.

tors for each separate grower. The evolution of systems of 10 to 15 interacting growers shows various classical cognitive growth phenomena, such as evolution toward a stable end point, transient regressions in some growers, stable regressions (some growers disappear), stepwise growth, and so on (see van Geert, in press).

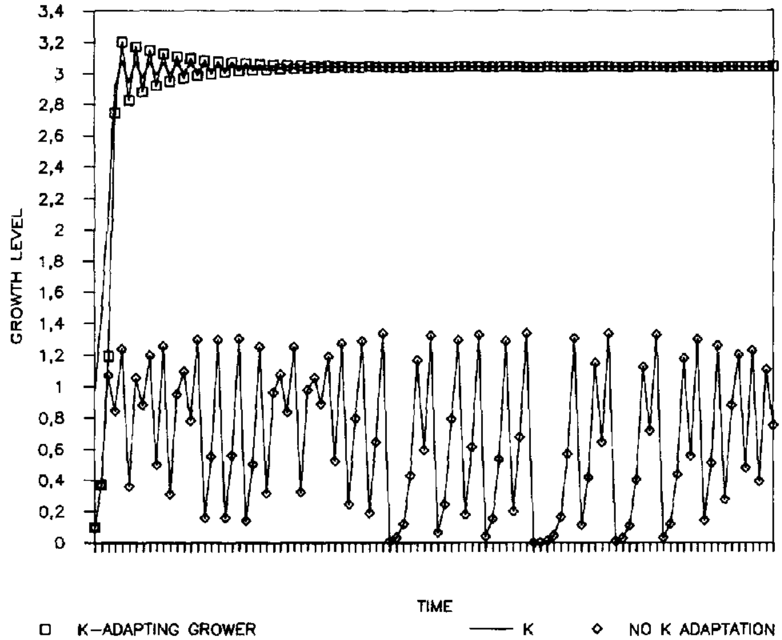

Figure 26. If carrying capacity K adapts to a rapid grower as a function of the unutilized opportunity for growth, $+U$, chaotic oscillations of the growth level L are damped, and K and L move toward a common point attractor at a higher level (top curve); if K does not adapt, L follows a chaotic course (bottom curve).

How Can Growth Phenomena Account for Stages?

Stages are traditionally accounted for in terms of structural models. Because an analysis of the way in which structural theories explain stages would far exceed the scope of this article, the reader is referred to other publications in which this work is undertaken (van Geert, 1986, 1987a, 1987b, 1987c, 1988a, 1988b). Note that the current concept of *stage* has shifted away from the classical Piagetian view of an overall state definition in terms of unique and characteristic structural features of the entire cognitive system, to a view that is much more content specific. Stages may occur within, and not necessarily across, cognitive domains or dimensions, and they may be characterized by various qualitative as well as quantitative properties (Levin, 1986). The question I address here is in how far the present growth model, which is explicitly gradualistic, may account for stage phenomena.

Logistic growth and stepwise shifts. Stepwise shifts in the magnitudes of characteristic variables are frequently seen as the major indication of stage shifts (Fischer & Canfield, 1986; see also Fischer, Pipp, & Bullock, 1984; Globerson, 1986), whereas a stage as such corresponds to a temporarily stable level (Fischer, 1983a). Stage shifts do not necessarily amount to the construction of entirely new skills, structures, and so on characteristic of the new stage. Those skills and structures were often present long before the onset of the stage shift, but they existed in germinal form (as *décalage* phenomenon, or as an innate or at least very early generic concept, a possibility even Piaget took seriously; Piaget, 1968). The germinal form may last for a rather long time before it starts to grow. Growth occurs in the form of a spurt followed by a leveling off toward a steady final state. This is exactly what the logistic growth model explains, and it does so in purely quantitative and gradualistic terms. In fact, what makes the difference between an apparently slow and quasi-linear increase in a variable and an almost quantum-leap-like emergence of the steady state of a variable is the height of the growth rate factor rather than some hidden structural factor, such as a restructuring of an underlying rule system causing the growth spurt. This underlying restructuring, if any, might be the result rather than the cause of a growth spurt. That is, a potential restructuring of the underlying rules (or whatever generative structure is assumed to cause performance) probably constitutes a response to the increasing pressure on the cognitive rule system that follows from the fact that the application of a rule or principle steadily grows and thus requires a more efficient or more powerful system than the one already available in order to be able to manage the significant increase in the domain of application.

Shifts and stages resulting from changes in resource and control variables. A quantum-leap-like shift in a major variable may amount to the effect of growth in either an underlying resource variable or a control variable. By *resource variable* I mean a variable that significantly contributes to the carrying capacity of a grower. For instance, the carrying capacity of complex problem-solving strategies is clearly dependent on the size and efficiency of working memory. In fact, this is the sort of hypothesis put forward by several neo-Piagetian researchers (e.g., Case, 1985; Pascual-Leone, 1970) who related Piagetian stage transitions to increase in working memory. By substituting different values for L and K in the difference equation of logistic growth (Equation 17a), one can demonstrate that the effect of stepwise increases in carrier capacity, for instance as a consequence of working memory growth, is significant only for growth variables that are close to their carrying capacity level, or their upper growth limit. This is so because the absolute speed of growth strongly reduces in the vicinity of K such that

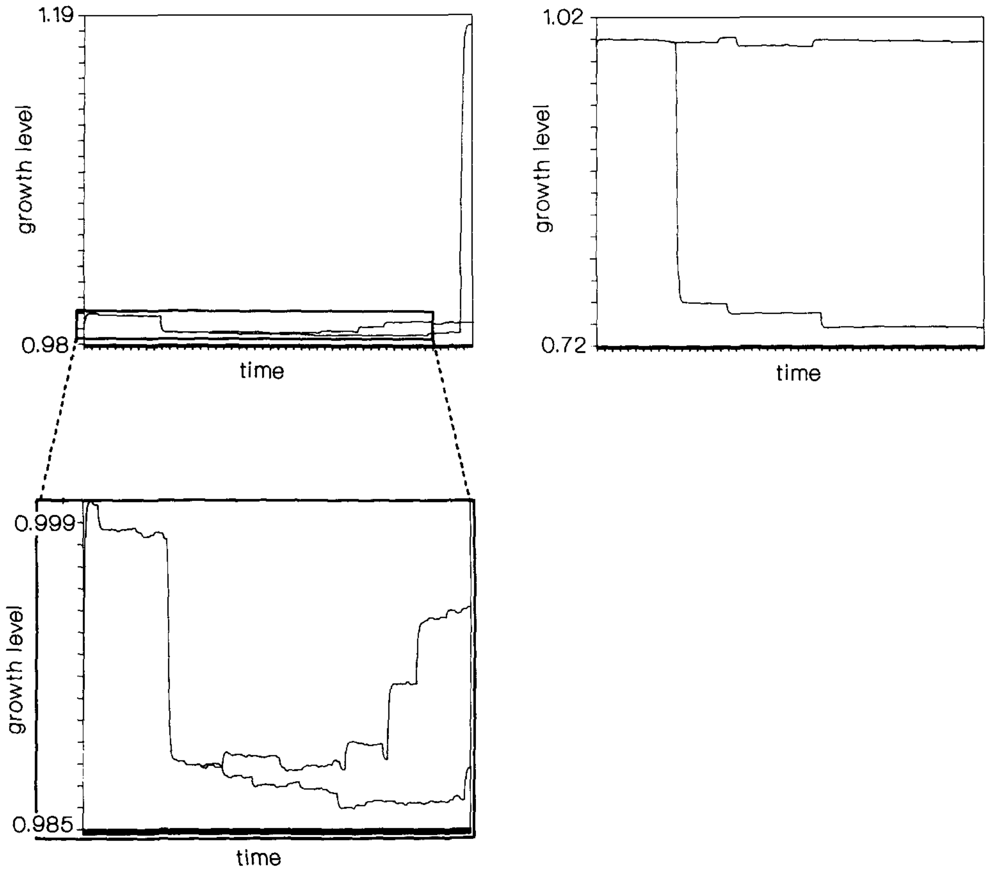

Figure 27. If carrying capacity adapts to growth level L as an inverse function of the unutilized capacity for growth, or as a function of $(-1/U)$, L and K follow a quasi-random meandering course (detail of left graph is shown at bottom of panel). (Extremely small initial state differences in the damping factor may cause dramatic differences in the course of the growers only after a considerably long interval; in some cases, unexpected drops or rises of about 25% may occur. Lines represent separate growers that differ only in the damping function. The left differs from the right only in the initial state condition.)

any sudden rise in K will suddenly increase the distance between K and L, which automatically results in a higher absolute growth speed expressed in the form of a jump. In this connection, a short discussion of *functional* and *optimal* levels of skill development, as they are termed by Fischer and Pipp (1984), may be relevant. The functional level, measured under conditions of low support, increases only slowly and more or less linearly. The optimal level, measured under optimal contexts (i.e., optimal practice and support) shows an S-shaped growth that is significantly higher than the functional level. What is probably witnessed in this case is the effect of different growth rates in the environmental help and support as expressed by the carrying capacity. If the growth rate of K is significantly lower than that of L (the growth level of some specific skill), L will very closely follow the slow, gradual increase of K in that L is always asymptotically close to its upper level as determined by K. Phenomenally, slow growth—although theoretically S shaped—appears in the form of a slow linear increase. If the help and support level grows much faster than the skill level itself—because the environment is particularly sensitive to the growth of the skill level in the subject, for example—L, provided its growth rate is high enough, will show the characteristic S shape of logistic growth and the high upper limit that is associated with high and adequate help and support. Quantum-like shifts may be the effect of a rise in a resource, control variable, or both. I interpret *control variable* (Fogel & Thelen, 1987; or *order parameter,* Haken, 1987) to mean any variable other than the carrying capacity and the growth rate that determines the growth in a dependent variable. In fact, a control parameter acts as a timing device for the dependent variable. That is, the dependent variable cannot start growing until the control variable starts growing or until the latter has reached some threshold level. A simple way to model a control system is the following: The growth rate of a dependent variable D has a very small initial state value and grows as a function of the growth level of a control variable C. In this dynamics, the growth rate of the control variable functions as a timer for the onset of a very quick growth process in the dependent variable (see Figure 28; see the Appendix for details). An example of a powerful control parameter for a variety of cognitive growth processes is given by Mounoud (1986) and concerns the ability to embed information in new contexts. Mounoud suggested that the sudden growth of cognitive skills such as reading, writing, and formal thinking around the age of 6 to 7 is made possible by the emergence of the general cognitive capacity to extract information from one context and embed it in another.

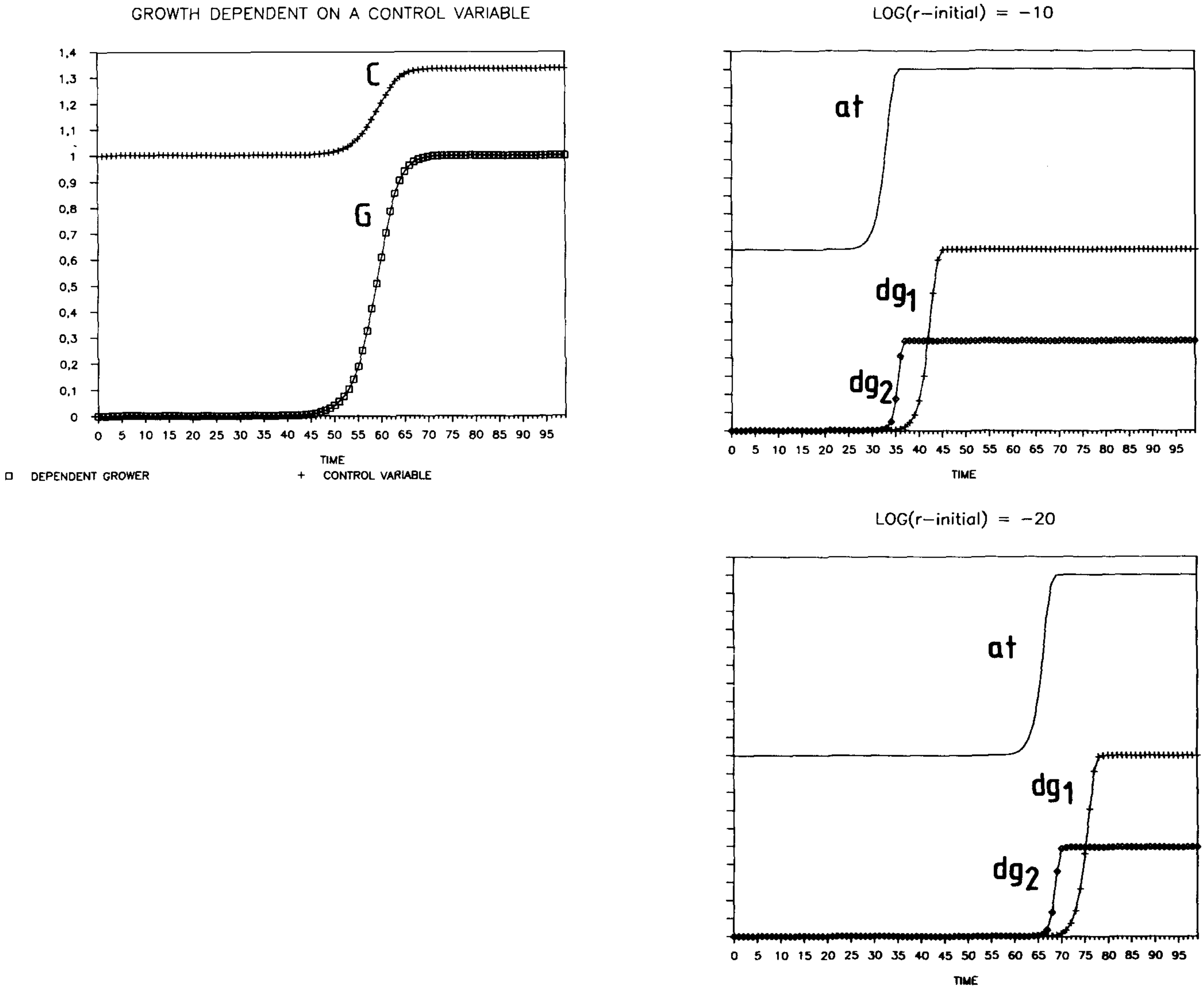

Figure 28. A Grower G, which seems to appear suddenly, grows as a consequence of growth in its control variable C (right). (If the control variable is an autocatalytic grower, its growth-onset time depends on its initial state level [top right vs. bottom right]; the autocatalytic timer [at] determines the growth onset of dependent growers [dg_1 and dg_2].)

Another example of a control parameter in a Piagetian model of cognitive development is the probability with which a specific sort of cognitive conflict will arise (e.g., a conflict between two opposing strategies for solving the same type of problem). If the growth of some specific form of logical understanding is indeed dependent on the probability that a specific sort of cognitive conflict arises and if this probability itself grows logistically, then the growth of the logical understanding will be timed by the underlying probability growth and take the form of a quantum-leap-like increase as shown in Figure 28 (top left). It is also possible to make an autocatalytic version of the timing device dynamics, that is, a dynamics where the timing of a sudden quantum leap is a function of the absolute growth of the dependent variable itself. The growth equation for such a dynamics is as follows:

$$r_{t+f} = (1 + e^{(L_t - L_{t-f})}) \cdot r_t - e^{(L_t - L_{t-f})} \cdot r_t^2 / K_r, \qquad (36)$$

which means that the growth rate of the growth rate of L is an exponential function of the absolute increase of L over the interval $(t - f)$ until t. It can be shown that in this dynamics, the duration of the initial state period is approximately an inverse logarithmic function of the magnitude of the initial state of the growth rate. The long initial state is then followed by a quantum leap in the growth level, which amounts almost to a sudden emergence of the variable at issue (see Figure 28, right). Autocatalytic timing functions like the present one might provide a model for cognitive capacities, the timing of which seems to be maturationally determined (i.e., they are apparently not dependent on experience or learning).

Stages and shifts resulting from different carrying capacities and mutual competition. A sequence of so-called conjunctive stages may result from separate growers with different growth rates. A conjunctive stage sequence (Van den Daele, 1968) can be defined as one in which each later stage encompasses the field of application of a former stage, in addition to new and more complex applications. In fact, conjunctive stage sequences amount to takeover phenomena. For instance, in Piaget's model, the formal operational logic takes over the domain of application of the concrete operational logic and adds to this the domain of specific formal operational applications. The sequence of such stages could result from an initial state in which all systems are present in a germinal state. The differences in growth rates and resource level account for the stagelike sequence. For the moment, assume that the growers entertain competitive relationships, which is quite likely anyhow, and that the positive feedback principle applies. Under these conditions, a temporary regression of a former stage may occur by the time it is surpassed by its successor (Figure 29; see the Appendix for details). Although the general shape of the interaction between the conjunctive growers is rather robust, the specific form, that is, whether and when regressions will occur, may rather strongly depend on small random factors (e.g., maximally 0.0001, as in the simulation from Figure 29). Takeovers and conjunctive growth are often, if not always, an indication of hierarchical relationships among growers. For instance, a growing skill *A* takes over the former domain of application of a skill *B* because *B* is a structural component of *A* or because *A* cannot start growing if *B* remains beneath a specific threshold level. For a growth model to explain long-term cognitive growth in various domains it should contain a model of such conditional relationships among growers. A model of conditional relationships requires an elaborated structural model of skills, rules, or knowledge describing their composition in terms of structural constituents.

Stages as shifts in a single grower. Finally, the concept of *stage* or *substage* may be applied to stepwise changes in a single variable (see Fischer, 1983b, for several examples). For instance, Dromi's (1986) vocabulary growth curve shows a temporary flattening which probably marks the transition to another substage during the one-word period. It is highly probable that the flattening of the learning curve following the onset of syntax is also only temporary and that it will be followed by a (probably slight) increase in the growth rate of new words, ending in a final leveling when the ultimate carrying capacity is approached. Corrigan's (1983) data on vocabulary growth in 3 children show a pattern of successive increases and decreases of the growth rate, resulting in a humped growth curve (see Figure 30). Such stepwise growth forms probably reflect the effect of oscillating growth in a sort of attentional resource variable. The time and effort a child may invest in specific learning (e.g., learning new words) is limited and could be controlled by some sort of activation-of-attention function attached to a specific grower (e.g., words or applications of a syntactic rule). This function actually determines the average amount of time and effort allocated to the learning process to which it is attached and thus specifies the child's interest in or motivation to perform some sort of acquisition task. It is also likely that this resource function is subject to positive feedback growth of the type discussed earlier—in other words, that it rises as a consequence of the success (progress and growth) of the dependent grower and that it tends to fall when some intrinsic resource limit (i.e., its carrying capacity) is crossed. This will, of course, negatively affect the growth rate in the dependent variable.

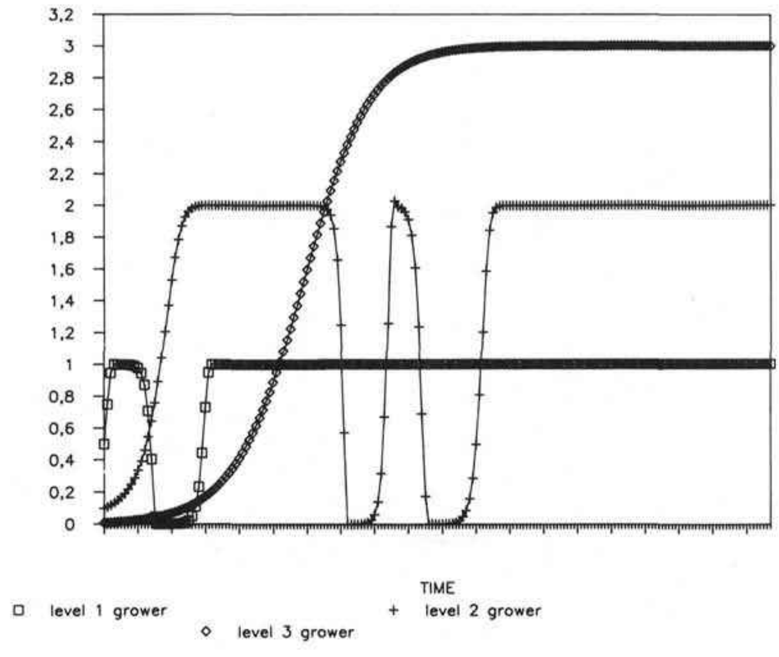

Figure 29. Three conjunctive growers with different growth rates produce a steplike sequence and temporary regression after takeover by a cognitively more powerful grower (e.g., sensorimotor, preoperational, and operational cognitive strategies).

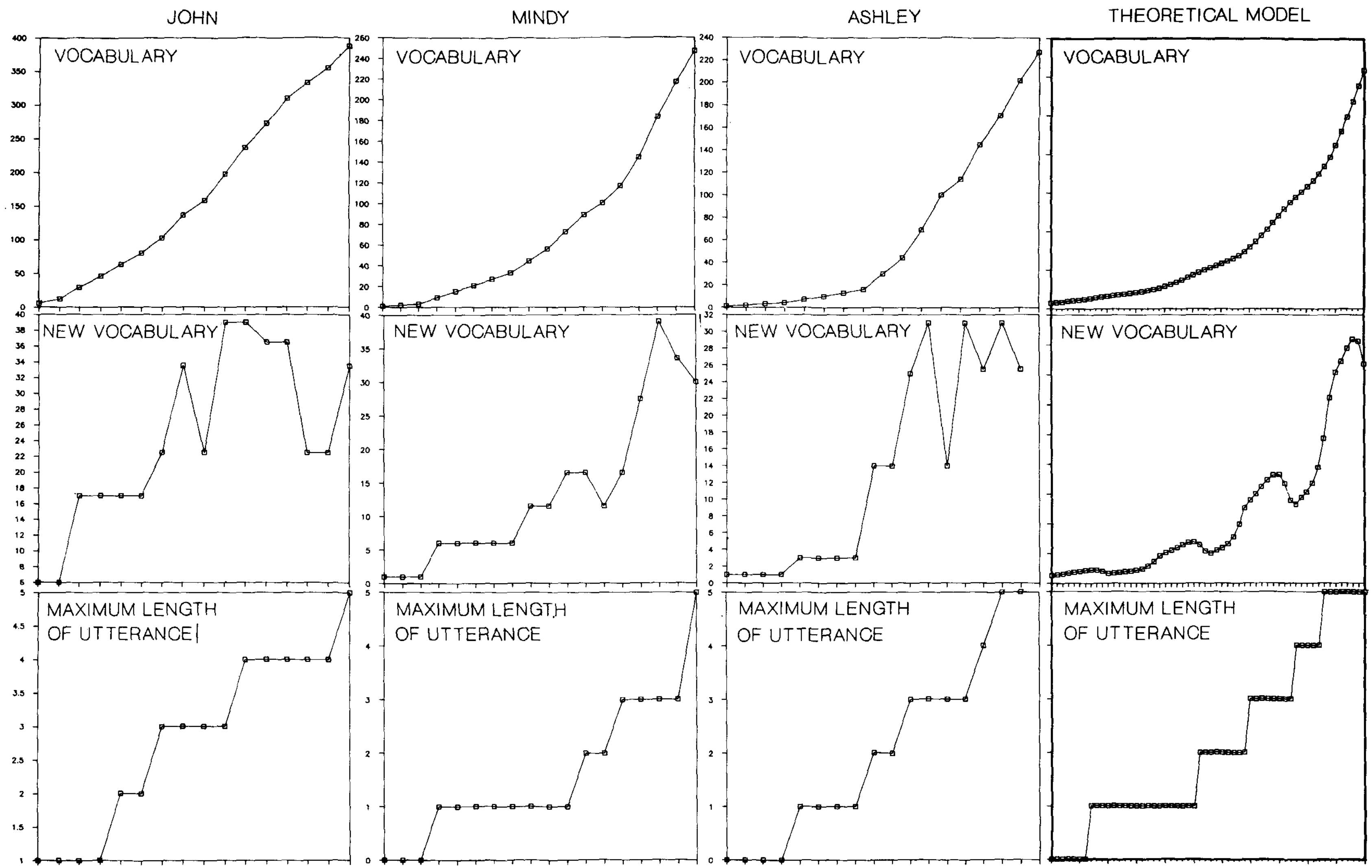

Figure 30. A mathematical simulation of empirical data on the growth of vocabulary and maximal length of utterance. (Horizontally, the diagrams show the increase in vocabulary [top], the number of new words acquired each month [middle] and the maximal length of utterance each month [bottom]. Vertically, the diagrams correspond to data from 3 children [John, Mindy, and Ashley] and simulated data based on the dynamics from Figure 31 [rightmost column]. Adapted from "The Development of Representational Skills," p. 60, by R. Corrigan, 1985, in K. Fischer, *Levels and Transitions in Children's Development,* San Francisco: Jossey-Bass. Copyright 1985 by Jossey-Bass. Adapted by permission.)

However, this fallback implies that more time and effort will become available to some other acquisition process.

For instance, one may assume that the child tends to temporarily invest much time and effort in learning new words and, as this investment decreases, that the time and effort not spent on word leaning will be invested in the growth of syntax expressed in the form of a growing mean length of utterance (MLU; assuming that this is a reasonable measure of overall early syntactic growth. Individual MLU data show that the increase is not smooth, but rather irregular; Brown, 1973, Pérez-Pereira & Castro, 1989).

The dynamics of this process is represented in Figure 31. In the present dynamics, two parameters are important. The first is the strength of the competition relation. If it is low, the growers grow smoothly and independently of one another to their proper maximum. The stronger the competition, the more steps occur and the more irregular they are. Mathematical implementation of a dynamics with rather strong competition leads to a picture of stepwise increasing growth curves, with mutually exclusive plateaus and rises in the growers that compete for the same attentional resource (see Figure 32; see the Appendix for details). A second important parameter is the amount of damping of the effect of the oscillating resource function. For instance, it is not necessarily the case that if the child's attention to syntactic aspects of language is temporarily minimal, syntactic growth simply stops (an implicit assumption made in the model from Figure 32). Rather, under such circumstances, growth rate decreases but is not reduced to about zero. This phenomenon may be used to model the Corrigan (1983) data mentioned earlier. Corrigan compared, among others, vocabulary growth with growth in maximal length of utterance in 3 children between 10 and 27 months of age. Her data can be mathematically modeled quite well by a dynamics of the sort represented in Figure 31, given that the effect of the oscillating resource variable is sufficiently damped (see Figure 30; see the Appendix for details). Again, this simulation is based only on a set of initial state values, which then develop deterministically in accordance with the equations based on the dynamics from Figure 31.

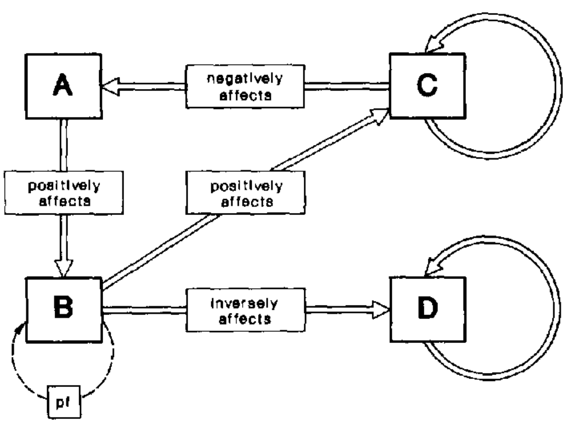

Figure 31. A model of a dynamics explaining long-term stepwise growth. (Two bootstrap growers *C* and *D* [circular arrows] grow antagonistically as a function of a resource *B*. Whereas *B* is positively affected by *B*, *D* is positively affected by the inverse of *B*; i.e., $K_B - L_B$ [hence, the qualification "inversely affects" on the *B–D* arrows]. *B* is subject to positive feedback [hence, the broken "pf" arrow] and is positively affected by a grower *A*. There is a competitive relationship from *C* to *A*; i.e., *C* negatively affects *A* and feeds upon the growth of a variable *A*.)

This discussion of how the growth model may account for stage phenomena has been restricted to quantitative aspects. Qualitative shifts, in the sense of the emergence of new forms, constitute in fact the major challenge to any developmental approach (Thelen, 1989). In principle, the quantitative approach presented here could contribute to the explanation of qualitative changes, for example by using the growth mechanism as a major transition factor in synergetics-type models. In such models, many variables cluster into simple structures that are characteristic of different developmental stages.

Some Final Prospects

The major idea behind the cognitive and language growth model discussed in this article is that cognitive growth occurs under the constraint of limited resources, with either mutual support or competition for resources among the cognitive growers that constitute a person's cognitive and language systems. This point of view explicitly subsumes cognitive growth under the general laws of thermodynamics: The cognitive system is a system carrying complex information and as such is far from thermodynamic equilibrium. Increasing its order and information load consumes time and energy. The cognitive system is also a self-organizing system, maintaining and increasing its own order, provided sufficient resources are available. Its form is to a great extent determined by the fact that these resources are limited.

I have shown that cognitive growth can be modeled mathematically in the form of a logistic difference equation, which applies to all—or at least to a very significant majority—of the variables involved in cognitive growth processes. All such variables interact and react with one another, thus making even relatively simple growth dynamics complex, transactional events. The growth curves resulting from such dynamics were often very difficult, if not impossible, to predict on the basis of simple linear extrapolation of initial state properties. A dynamic systems approach like the present one might change the meaning—at least the connotational meaning—of concepts such as *deterministic, random, predictable,* and so on. For instance, in several cases, growth sequences appear very similar to random sequences, although they behave completely differently from real random sequences, for instance in that under specific circumstances they may evolve toward stable points, which random sequences will never do. Although the equations involved are very easy to understand and involve no complicated mathematics, they are very difficult to solve; that is, it is very difficult to give general answers to questions such as under which conditions specific equations will lead to stable solutions, to damping of oscillations, and so on. Thus, further mathematical scrutiny of the equations presented here is necessary.

I have discussed several dynamics without entering deeply into the psychological and process interpretations of these dynamics and their components. The major aim of this article was to show that dynamics of the sort I have discussed may provide

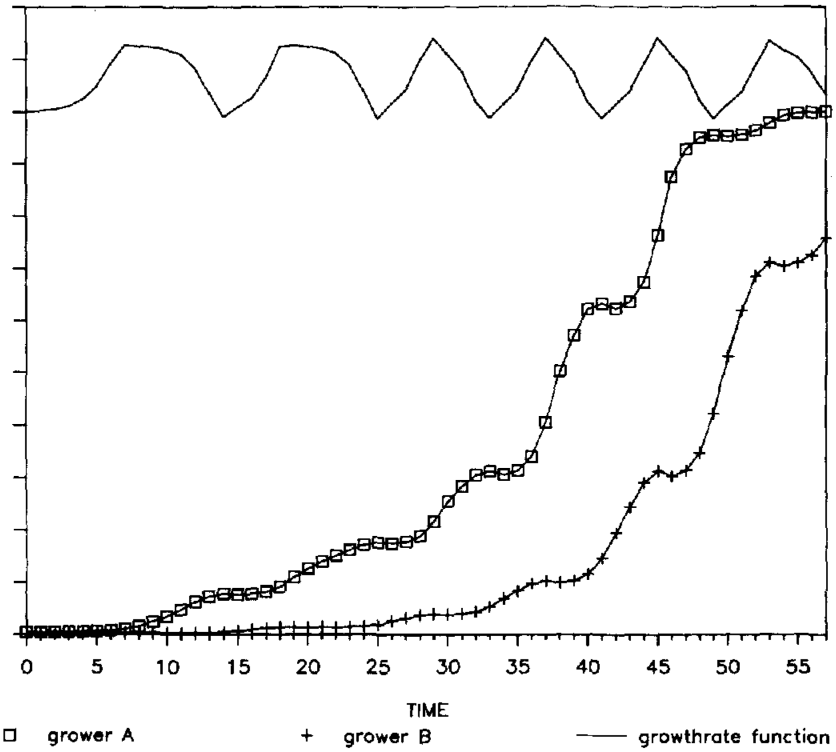

Figure 32. Growth curves following a stepwise course, based on the step dynamics from Figure 31. (Both growers correspond to the components *C* and *D* in the dynamics from Figure 31, and the oscillating resource corresponds to the *B* component. The *A* component from Figure 31 is an intermediary variable not represented in this figure.)

a plausible general model of several cognitive growth processes. The real work of thoroughly testing the empirical merits of specific dynamics applied to specific developmental fields and of finding psychological interpretations that can stand up to empirical scrutiny is still to be done. Moreover, several obvious theoretical questions have not been answered. I give two examples. One concerns the application of mixed interactions, that is, interactions in which one affects another positively whereas the other affects the first negatively. An example of mixed interaction is the negative support dynamics, such as paternal correction following the growth of unwanted skills or habits. Another question still to be answered concerns the simulation of interactions in more complicated systems of cognitive growers, for instance, systems consisting of 10 or more growers which interact with each other, the effects of random perturbations on the long-term course of processes, and so on.

From a methodological point of view, applying the present dynamics model would ideally require longitudinal individual studies with a sufficiently dense measurement schedule and with maximally reliable data. This is especially so in cases where growth functions show strong interindividual variability. Unfortunately, this requirement is much more than most current studies in cognitive development can offer. If individual growth functions do not differ too considerably in general shape and rate, cross-sectional and mixed designs may be used to test several growth patterns. For instance, in a mixed design with at least two consecutive measurements for each age group, one may use differences in the slopes of the curves or the degree of instability of measurements over time as indicators of an underlying growth form. Cross-sectional group data should be used selectively. For instance, if group data reveal a transient regression, then such regression should be characteristic of a significant portion of the individual growth curves, if such curves would be available (and provided the groups developed under largely similar cultural and social–historical cohort conditions). If cross-sectional group data fail to show regressions, this may be due either to the fact that individual regressions have compensated one another or to the fact that no such regressions have occurred. Methods in which multiple tasks are presented to children and related to specific developmental functions may also be used to reveal different underlying growth processes (Fischer et al., 1984).

Another methodological aspect of the model is that intraindividual instability of growth data is not considered a weakness or a sign of unreliable measurement, but rather as an essential characteristic of cognitive growth. Of course, reliable measurement remains an essential condition for theory building. On the other hand, one should not automatically imply stability over time as a criterion of reliability of measurement, because there are many forms of growth conceivable in which temporal instability is a major structural characteristic of growth instead of some sort of aberration.

As far as the underlying assumptions of the logistic growth model and the dynamic systems approach apply to microgenetic events, such as the growth of attention or skill during a single experimental session, the model may also be used to describe short-term changes in behavior or short-term learning effects. Finally, in this dynamics model, the data from a group of subjects on a specific form of growth (e.g., of inversion rules in *Wh*-questions) should not be considered for means and overall data but should be viewed as a collection of trajectories specifying a state space. The developmental or growth model should provide a general model of this state space and explain which individual trajectories are theoretically possible and which are not. My approach does not start from the idea that in each process of development some orthogenetic line of development should exist that is typical of a group of subjects as a whole. Rather, the dynamics model describes cognitive and language growth as a constrained bundle of individual growth possibilities or trajectories that proceed as a result of a specific underlying dynamics.

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(Appendix follows on next page)

Appendix

Equations, Initial State Values, and Constants Used in Growth Sequences Depicted in Figures

All growth sequences depicted in the figures are described, except those for which a description is given in the main text. The mathematical symbols correspond to those in the equations mentioned. The variables include real variables, meaning those changing during the growth process, and constants. Of the real variables, only the initial state values are given. If the equations used do not correspond to equations explicitly discussed in the text, they are presented in this appendix. The difference form of the logistic growth equation is either presented by its equation number (17a), or presented in abridged form—that is, with the symbol f followed by its arguments, which are separated by a comma. The arguments are presented in a fixed order: L-r-K. For instance, $K = f(K, d \cdot L, K_K)$ means that the growth of a carrying capacity K is a logistic difference function of the level of K occurring a time interval f ago, a growth rate that is a product of a damping function d and the growth level of a grower L (also a time interval f ago), and finally a carrying capacity that corresponds to the highest possible stable level of K.

Figure 12: Competition Without Positive Feedback

Initial State Variables

$r_A = 1.25$

$L_A = 0.5$

$r_B = 0.35$

$L_B = 0.01$

Constants

$K_{r_A} = 4$

$K_{r_B} = 2$

$c_A = 0.2$ competition factor damping the effect of $(K_A - L_B)$

$c_B = 0.1$ competition factor damping the effect of $(K_B - L_A)$

Equations

28a–f

Figure 13 (Top Figure)

Initial State Variables

$r_A = 0.8$

$L_A = 0.5$

$r_B = 0.07$ (and 0.1, 0.24, and 0.33 for the other examples in Figure 13)

$L_B = 0.1$

Constants

$K_A = 1$

$K_B = 2$

$K_{r_A} = 2$

$K_{r_B} = 2$

$c_A = 0.1$

$c_B = 0.05$

Equations

28b–e

29a–b

$$r_{A_{t+f}} = \text{abs}[(1 + (K_{A_t} - L_{B_t})/c_A) \cdot \text{abs } r_{A_t} - r^2_{A_t} \cdot (K_{A_t} - L_{B_t})/(K_{r_A} \cdot c_A)] \cdot (L_{A_t} - L_{A_{t-f}})/\text{abs}(L_{A_{t-f}})$$

Figure 15: Competition Among Alternative Strategies

Initial State Variables

Figure 15a:

$K_A = K_B = 1$

$r_A = r_B = 0.4$

$L_A = L_B = 0.01$

$c = -0.05$

$c' = -0.15$

Figure 15b:

$K_A = K_B = 1$

$r_A = r_B = 0.05$

$L_A = L_B = 0.01$

$c = -0.1$

$c' = 0.099$

Figure 15c:

$K_A = K_B = 1$

$r_A = r_B = 0.05$

$L_A = 0.0125$

$L_B = 0.01$

$c = -0.1$

$c' = -0.09$

Figure 15d:

$K_A = K_B = 1$

$r_A = r_B = 0.05$

$L_A = 0.0126$

$L_B = 0.01$

$c = -0.1$

$c' = -0.09$

Equations

30a–b

Figure 19: Effect of Sensitivity Levels

Initial State Variables

$K = 0.2$

$r = 3$

$L = 0.02$

Insensitive: $d_K = 5$

Sensitive: $d_K = 1.1$

Inconsistent 1: $d_K = 0.017123$

Inconsistent 2: $d_K = 0.0171$

Constants

$M_K = 0, 2$ (minimal structural growth level M of K is 0.2)

$M_L = 0.01$ (minimal structural growth level M of L is 0.01)

$d_r = 0.5$

$K_r = 0.2$

$K_K = 1$

Equations

31
32a–g
for 32f, $r_{r_{t+f}} = \sin(L_t) \cdot d_r$ (the growth rate of the growth rate is a sinus function of the growth level of L times the damping function of r)

If $K < 0.2$, reset K to 0.2

If $K > 2$, reset K to 2

If $L < 0.01$, reset L to 0.01

Figure 20: Nonlinear Effect of Sensitivity on Final State

Initial State Variables

$K = 0.4$

$r = 3$

$L = 0.02$

$A: 0.01 < d_K < 20.01$

$B: 0.01 < d_K < 2.01$

$C: 0.65 < d_K < 0.85$

$D: 0.0704 < d_K < 0.0714$

Constants

$M_K = 0.4$

$M_L = 0.01$

$d_r = 0.5$

$K_r = 0.1$

$K_K = 1$

Equations

(as in Figure 19)

Figure 22: Simulation of Inversion in *Wh-* Questions

Initial State Variables

$K = 0.2$

$r = 4.7$

$L = 0.165281$ (similar to the initial state value, with 100% correct set to 1)

Constants

$K_K = 1$

$K_r = 0.132$

$d_r = 0.025$

$d_K = 1.0526$

Equations

32b, 32d, 32e, 32g
33

for $r_{K_{ef_{t+f}}} = (\cos L_t - 0.4) \cdot d_K$ in Equation 33

and $r_{r_{ef_t}} = \sin L_t \cdot d_r$ in Equation 32a′,

$r_{ef} = f(r_{ef}, r_r, K_r) + 0.4$

Figure 24: Growth of the Present Progressive Form

Correct Strategy

Initial State Variables

$K_{ef} = 0.03$

$r_{ef} = 9.75$

$L = 0.00065$

Constants

$$K_K = 1$$
$$K_r = 0.5$$
$$r_K = 0.05$$
$$r_r = 0.01$$
$$M_K = 0.001$$
$$M_L = 0.001$$

Equations

$$L = f(L, r_{ef}, K_{ef})$$
$$K_{ef} = f(K_{ef}, r_K, K_K)$$
$$r_{ef} = f(r_{ef}, r_r, K_r)$$

If $K_{ef} < M_K$, reset K_{ef} to M_K

If $L < M_L$, reset L to M_L

Wrong Strategy

Initial State Variables

$$K_{ef} = 0.03$$
$$r_{ef} = 0.6$$
$$L = 0.0035$$
$$r_K = 2.1$$

Constants

$$r_r = 0.05$$
$$K_r = 0.5$$
$$M_K = 0.001$$
$$d_{K_K} = 0.2$$
$$r_{r_K} = 0.05$$

Equations

$$K_{ef} = f(K_{ef}, r_K, K_K)$$
$$r_{ef} = f(r_{ef}, r_r, K_r)$$
$$M_L = 0.4/t$$
$$K_{K_{t+f}} = (K_{K_{correct_t}} - L_{correct_t}) \cdot d_{K_K}$$
$$r_{K_{t+f}} = \text{abs}[(1 + r_{r_K}) \cdot r_{K_t} - r_{r_K} \cdot r_K^2/0.5] \cdot (K_t - K_{t-f})/\text{abs}(K_t - K_{t-f})$$

If $K_{ef} < M_K$, reset K_{ef} to M_K

If $L < M_L$, reset L to M_L

Figure 25: Adaptation to Slow Growth

Initial State Variables

$$K = 1$$
$$L = 0.05$$

Constants

$$M_K = 0.05$$
$$M_L = 0.01$$
$$r = 0.05$$
$$d = -0.01$$

Equations

34a

Figure 26: Adaptation to Fast Growth

Initial State Variables

$$K = 1$$
$$L = 0.3$$

Constants

$$r = 3$$
$$M_K = 0.05$$
$$M_L = 0.01$$
$$d = 1.06$$

Equations

34a

If $K < M_K$, reset K to M_K

If $L < M_L$, reset L to M_L

Figure 27 Quasi-Random Meandering

Initial State Variables

$$K = 1$$
$$L = 0.998$$

Constants

$r = 0.3$

$d_1 = 1E - 10$ (10 to the power -10)

$d_2 = (1 + 1E - 13)E - 10$ [equals $(10 + 10^{-13})^{-10}$]

Equations

35a

Figure 28: Autocatalytic Timer and Dependent Growers

Initial State Variables

$$r_A = 1E - 20$$
$$L_A = 0.001$$
$$L_B = 0.001$$

Constants

$$K_A = K_B = 1$$
$$d_{r_B} = 11$$

Equations

36

$$r_{B_t} = (r_{A_t} - r_{A_{t-f}}) \cdot d_{r_B}$$
$$L_A = f(L_A, r_A, K_A)$$
$$L_B = f(L_B, r_B, K_B)$$

Figure 29: Three Conjunctive Growers

Initial State Variables

$$L_1 = 0.5$$
$$L_2 = 0.1$$
$$L_3 = 0.01$$
$$r_1 = 1$$
$$r_2 = 0, 1$$

$0 < p < .001$ (p is a positive random number)

Constants

$$d_1 = 0.1$$
$$d_2 = 0.04$$
$$r_3 = 0.1$$

Equations

17a for $L_{1,2,3}$
29a–b for $r_{1,2}$

Figure 30: Simulation of Corrigan's data

Initial State Variables

$L_{voc} = 6$, where voc = vocabulary

$L_{MLU} = 0.1$, where MLU = maximal length of utterance

$$r = 0.001$$
$$r_r = 0.01$$

Constants

$$K_{voc} = 5{,}000$$
$$K_{MLU} = 10$$
$$K_r = 0.4$$
$$K_{r_r} = 1.5$$
$$d_{r_r} = 1$$
$$d_r = 0.083$$

Equations

$$L_{MLU} = f(L_{MLU}, (1 - r) \cdot d_r, K_{MLU})$$
$$L_{voc} = f(L_{voc}, (0.05 + r/10), K_{voc})$$

29a for r_r, provided that

$$r_{r_r} = [\cos(L_{voc}/K_{voc}) - 0.4] \cdot d_{r_r}$$
$$v_{t+f} = (r_{t+f} - r_t)/|(r_{t+f} - r_t)|.$$

L_{MLU} is rounded off to whole numbers

Figure 32: Stepwise Growth

Initial State Variables

$$r = 0.01$$
$$r_r = 1$$
$$L_A = 0.01$$
$$L_B = 0.001$$
$$K_A = 0.2$$
$$K_B = 0.1$$

Constants

$$d_{r_r} = 0.166$$
$$K_{r_r} = 2$$
$$d_A = 0.6$$
$$d_B = 0.4$$
$$d_{K_A} = 0.3$$
$$d_{K_B} = 1.5$$
$$K_r = 1$$
$$K_{K_A} = K_{K_B} = 2$$

Equations

17a for $L_{A,B}$
29a for

$$r_{r_r} = (1 - L_A - L_B) \cdot d_{r_r}$$
$$v_{t+f} = d(r_{t+f} - r_t)/|(r_{t+f} - r_t)|$$
$$K_A = f(K_A, d_A \cdot L_A, K_{K_A})$$
$$K_B = f(K_B, d_B \cdot L_B, K_{K_B})$$

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