V
UNDERSTANDING CHILDREN

For many years knowledgeable Americans have been intrigued by the way Russian culture deals with children. In this section we present a personal note, written by an important Russian mathematician, that catches many key aspects of the Russian approach. Although this is written in simple and unassuming language, in fact one could build an entire university course on the explication of the key themes that are suggested here by Alexander Zvonkin.

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Mathematics for Little Ones

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I am sure that all of you have seen such episodes many times: a young mother hides behind a curtain, then peeps out with a smile and a "peek-a-boo", then she hides again. And her tiny baby greets each emergence clapping hands and shrieking with joy . . . . Both are quite happy. Of course, it never occurs to them that what they are truly engaged in is called *mathematics*.

I have written the above phrase not to shock the reader or to catch him with a hook of a far-fetched paradox. I am in earnest. If you read psychological books, you will learn that at the age between birth and 18 months one important intellectual task which a baby has to master is to discover *the law of the constancy of objects*. That is, that things do not disappear when we cease to see them, but on the contrary usually continue to exist. Watch babies of this age. A boy pulls a ball from under the table. A few minutes later the ball again rolls away somewhere—this time, under the armchair. The boy can even see it there, but he crawls to fetch it from under the table, because it was the action that has just been successful. A girl of one year has come with her parents to visit friends. Her father leaves the flat to have a smoke in the hall. The daughter goes with him as far as the entrance door and when it closes, she cries and runs to look for him in the opposite direction: into the room where they had been together a few minutes earlier. One must accumulate immense experience to understand that the father who had gone to the hall will most likely come back also from the hall. The game of "peek-a-boo" adds a tiny bit to this experience: it turns out that such an important object as mother, having disappeared behind the curtain, still continues to exist somewhere near and soon reappears from behind the same curtain.

Children grow, and their picture of the world becomes more adequate. A boy of two comes up to his father early in the morning and touches his shoulder, "Are you sleeping, Dad?" "No," answers Dad, "I am not sleeping. I am having tea in the kitchen." The boy is extremely amazed, but he goes off to check—just in case. In a moment he comes running back with a joyful shout: "Hey, you are not in the kitchen! You are here!" A short, but excellent lesson in mathematics. It made the child subject to doubt the law he had discovered earlier, to undertake research of his own and get even more certain in his knowledge.

All these situations (and the like) were to me a kind of starting point or a

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guiding star when I thought what mathematics we may and should choose when dealing with pre-school children. Certainly, each age requires its own mathematics, but it must always be *the mathematics the child is engaged in on his own, without our help*. The part which the adult should be limited to is to formulate more accurately the questions that arise in the child's everyday experience, to single out and to discuss together the possible answers, and to gingerly direct the child's own reflections along certain lines.

To my mind, the conceptions of what and how we should teach to pre-school children are generally fraught with numerous misunderstandings and even blunders. To the question *what* we should teach, the traditional answer is: mainly arithmetic and a bit of geometry. As to *how*, the basic idea here is expressed by the following words: to teach, to explain, to show, to demonstrate, to repeat, to consolidate, etc. This idea repels me most of all. When I read claims that nowadays the mathematical requirements for children in kindergarten have greatly increased, I feel upset because the "increased requirements" look so extremely boring. As for arithmetic, nobody would argue that the skill of counting is very useful. But what does this skill mean?

Let us imagine we are children who try to learn arithmetic—though in Japanese. Here are the digits for you: *iti, ni, san, si, go, roku, siti, hati, ku, ju*. I just wonder how much it will take you at least to learn the sequence by heart? When you have coped with the task, try to count backwards, from *ju* to *iti*. Now, if you can already do this, let's try to do simple sums. Answer without delay and, if possible, without mentally translating into English: how much is *roku* and *san?* Subtract *go* from *siti*. Now divide *hati* by *si*. Here is a problem for you: mother bought *ku* apples at the market and gave *ni* apples to each of *si* children; how many apples remain? (All your answers must also be in Japanese.) If after a month of thorough practice you have mastered this difficult skill and learned quick calculations up to *ju*, accept my congratulations: you have an excellent drill memory. And, it goes without saying, all that has very little to do with your intelligence.

There are mathematical difficulties proper in calculations as well. But more often than not they remain behind the stage, invisible and unnoticed. And this is, perhaps, only for the better. Otherwise, the enthusiasts of early teaching would have rushed to explain to the child what he is yet unable to understand, wishing to pull him by the collar up to the next step. *And he could have done it on his own*. For many years, I have carried out a kind of experiment: during all pre-school years I never taught my son to calculate (and he didn't go to the kindergarten). By the time he went to school he didn't in any way fall behind the children of his age and, even surpassed many of them, but we parents never took *this* to be his principal intellectual achievement.

The second traditional subject of pre-school mathematics is geometry. It is assumed that children are to be given certain information concerning geometrical taught the simplest measuring techniques. (By the way, there is an excellent book by V.G. Zhitomirsky and L.N. Shevrin, *Geometry for Little Ones*, highly readable and brilliantly illustrated by A. Golovchenko that can be strongly recommended both to parents and children.) But just think: if a child easily tells a spoon from a fork, why should it be difficult for him to tell a square from a triangle? To be sure, this is not difficult. What he has trouble with is comprehending logical links (interrelations) between the notions, as well as the operations that are allowed with these objects. For example, I met the beginners who thought that if a square is drawn askew, it ceases to be a square but becomes a quadrangle. Moreover, the question whether there are more squares or quadrangles requires exceptional logic.

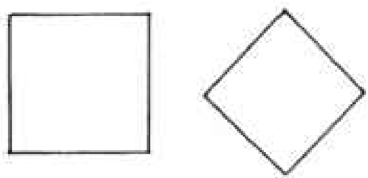


Figure 1.

In brief our task is not to impart knowledge to the child but rather to provide the child with material for reasoning and observations. If the whole situation is viewed from this angle, triangles and squares immediately lose their primogeniture: problems about spoons and forks are no less mathematical if they *provide a chance to think*. Perhaps pre-school mathematics deals with numbers and figures only because they are present in school mathematics as well? Isn't it just a tribute to the tradition? We can tell the little ones very few meaningful things about these objects. Cannot the problem be viewed more broadly?

It is easy to criticize other people. But what can I propose instead? Is there any other way?

When my son Dima was four, I couldn't wait any longer and organized a real mathematical circle. We had our sessions once a week, roughly for half an hour. There were four participants: Dima, Gene, Pete, and Andrew. Dima was the youngest; the eldest, Andrew, would soon be five. All the boys were our neighbors, Dima's playmates. Soon I started a diary where I recorded our sessions: both failures and achievements. But, as it often happens, it is the first session that I remember most clearly.

We sit down around a coffee table. Naturally, I am nervous. First, I tell the children that we are going to study mathematics and, to encourage myself, I add that mathematics is the most interesting science in the world. Immediately comes the question, "What is science?" Now I have to explain: science is when people think a lot. Andrew is somewhat disappointed, "I thought there would be tricks". He had been told at home that Uncle Sasha would play mathematics and there would be tricks. "We'll have tricks too", I answer and curtailing the introduction, come to the point. This is the first problem. I put eight buttons on the table. The boys do not wait for my instructions but all together immediately start counting them: so far "mathematics" and "counting" are synonyms to them. When the noise calms down, I can formulate the problem: "Now put *as many* coins on the table". So there are eight coins on the table, too. We place coins and buttons in two similar lines, one opposite the other. "What is more here, buttons or coins?" The children look at me somewhat puzzled; it takes them some time to formulate the answer, "Nothing is more". "So, the lines are equal", I say. "Now watch what I'm going to do." And I move the coins apart from one another, making their line longer. "What is there more of now?" "More coins!" shout the boys in chorus. I suggest that Pete count the buttons. Though we have already counted them four times, Pete is not surprised by my task and counts them for the fifth time: "Eight". Now I tell Dima to count coins, he counts and says: "Also eight".

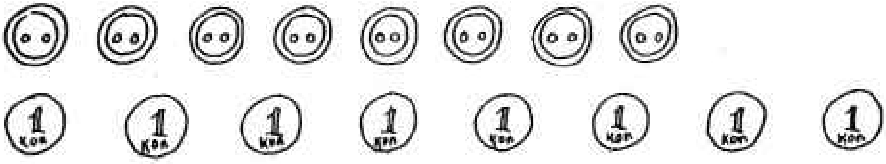


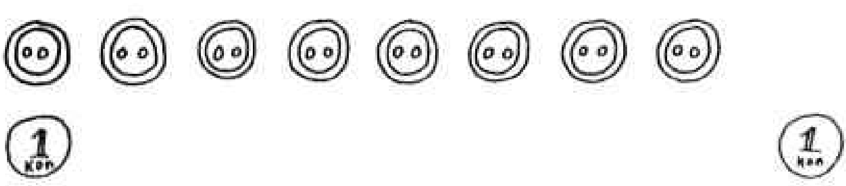
Figure 2.

"*Also eight?*" I emphasize the question by the tone. "So, their number is equal?" "No, there are more coins", assert the boys resolutely.

To tell you the truth, I knew beforehand that their answer would be like that. This problem is only one in the infinite series of problems that the great Swiss psychologist Jean Piaget gave to his children—subjects in the experiments. He came to the conclusion that little children do not understand things which seem self-evident to us (e.g., if several objects are rearranged or moved from one place to another, their number will not change in any way). I recall a typical episode. We had visitors, and we were short of one chair. Dima suggested that the guests be seated in another way: Uncle George should sit here, Aunt Suzy here, etc. He was surprised to see that he again lacked one chair. He suggested the third version, then the fourth one; every time, he was extremely surprised that one of the guests again had no place. Later I made of this episode a problem for the circle. I knew beforehand what the children would say. I knew, but somehow I didn't prepare a reasonable response to their answer. What would you do, reader? What would you say to the children?

The most widely spread mistake that almost all adults make is to begin explaining everything to children. "How could that be?" says the adult with a feigned surprise. "How could their number increase? We haven't added any new coins, have we? We have only moved them apart, that's all. The number of buttons and coins remains equal." This won't do. First, do not hope that your logic will convince a child: logical speculations do not seem convincing to those who cannot as yet reason logically. The only convincing factor is your intonation. And this will show the child that he again didn't rise to the occasion and did something in the wrong way. Grown-ups in general make a lot of inexplicable demands on children: for some reason one can't draw on the wall, or must go to bed when one is not sleepy or must not ask: "When will the uncle go away?" Now, too, from the viewpoint of a child the situation is very similar: though I see that there are more coins than buttons, I must for some reason say that their number is equal. (Hans Freudental, the Dutch mathematician, writes that children of 10 or 11 asked a new teacher whether he would demand, as the previous teacher did, that they change the sign when carrying the number from the left-hand part of the equality into its right-hand part. This requirement had to them the same meaning as the requirement to begin writing four squares to the right from the edge of the page.) Children do not give up very easily, they have healthy spirits. However, if you press them mercilessly you will achieve the situation when they don't act on the basis of their own views but will try to guess what answer the grown-up expects from them. But this is not our goal.

So, how should one behave? Well, first of all, you can exchange views: "What do you think, Gene? Do you agree, Pete? Why? *How many* coins are there more?" You may as well voice your opinion, but very cautiously and unobtrusively, with all kinds of reservations like "I believe," or "perhaps". You should use all your grown-up authority not to attach to it the absolute power of the only true judgement but to persuade the child that his own search and efforts are valuable and important. It is even more interesting *to make him see the contradictions* in his own views. "How many coins must we *take away*, to make the number of buttons and coins equal again?" "You must take away two." We take away two coins and count: there are eight buttons and six coins. "What is there more now?" "Equal." Fine. I again move coins wider apart and ask the same question. Now it turns out that six coins are more than eight buttons. "How has it happened?" "Because you've moved them apart." We take away two more coins, then two more. At last the situation is like this. Here a heated discussion starts. Some think, as before, that there are more coins than buttons, others have suddenly "seen" that there are more buttons. It seems to be the best moment to skip the problem and pass to another one: let them now think for themselves.



All these ideas did not come at once; actually here my mind drifted to some later events in my story—my own future speculations and future sessions. We had a similar problem in different clothes many times. For example, we had two armies, none of which could win because they had an equal number of soldiers. Then the soldiers in one army moved apart; their number increased and they began to win. The soldiers of the other army moved even wider apart, etc. (You can finish the story in accord with your own imagination.) We also had a problem in which the fox and the cat tried to deceive poor Pinocchio, moving apart five golden pieces and saying that their number thus increased. I learned not to expect easy victories. All the same, the children won't be able to master the law of the object quantity preservation sooner than in two or three years, no matter how you teach them. Besides, *the most important thing is they shouldn't do this.* The premature instruction is no more useful than premature birth. Every vegetable has its season, and we shouldn't forerun the natural course of things, in the field of intelligence as well. (This viewpoint is formulated here in a somewhat demagogical way only due to the lack of space. I am prepared to prove it proceeding from my own experience, from the authority of the most shrewd teachers and scholars and from the data of psychological experiments.)

But I repeat again, all those ideas came later. As to the first session, I am glad that some insight withheld me from explanations, and I merely passed to the next problem.

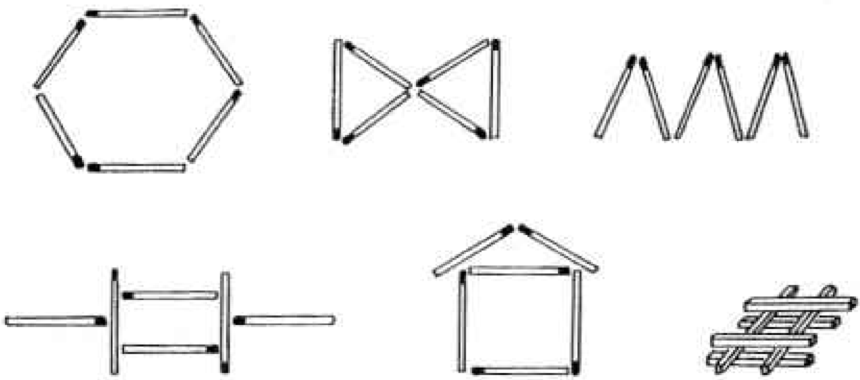


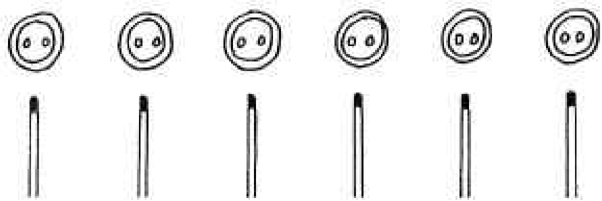
Figure 4.

I put six matches on the table and one by one composed various figures of them. Then I asked the children to count the matches in every new figure. And each time their number proved to be six. But no! My scholastic speculations have made my style too formal and dry. Let us return to the real children and see how it al happens *in vivo*. Each new result of counting is accompanied by an outburst of laughter and delight. Andrew and Gene shout there will always be six. Dima, [illegible] the matches out of my hands—he wants to make a fanciful figure himself. Pete, on the contrary, asks me very politely whether I can give him a few more matches. In a second their delight may turn into an uncontrolled fit of disorder. I must stop them somehow, and listen attentively to Gene and Andrew ("Why do you think so?") and keep in mind the new turns of their thoughts: Dima has just composed a three-dimensional figure, a *well*. I call the children's attention to it. Now even Andrew and Gene are not altogether certain the number will again be six. It is difficult to count the matches since the well collapses all the time. At last Dima has succeeded—he gets seven! Everybody is somewhat puzzled but not too much: let it be seven, though it is a bit odd. Well, my pedagogical task is not to tell the children the final truths but to awaken their curiosity. If any of the boys in a few days (or months) will suddenly build a well on his own and count the matches—just because he got interested and wished to learn how many matches there actually were—then my teaching method will reach its peak: this will be a small independent research! But if this doesn't happen, let us hope it will happen another time, with another problem. (Later I had many chances to see that this is what happened a lot of times.) Anyhow, I only say "how very interesting!" or "remarkable!" in the hope that the situation will *stick better in their memories*.

What an amazing thing is a child's memory! I can't resist the temptation to recall here a later event. At one of our sessions, (to be more accurate, at the 22nd) we discussed the question of the following kind: what is there more—animals or rabbits? Geese or birds? Men or people? Flies or insects? The questions seem to be somewhat monotonous, but the answers are strikingly various. For example, Pete believes that there are more flies than insects, because flies fly everywhere, while insects don't; besides, not all insects can fly. By the way, he is quite aware of the fact (I took the trouble to make sure!) that flies are also insects. Andrew is prone to think that there are more birds than geese since there are lots of birds in the world. I start telling him how many geese there are in the world, and he begins to hesitate. Gene and Dima give correct answers, but their explanations make me suspect that they too have not yet completely realized the great law, "the whole is larger than one of its parts". The next problem proved my suspicions were correct. I put on the table three cardboard figures. We discuss them in detail one by one and all three together. Each figure has four corners. So we can call each of them a quadrangle. Thus we have three quadrangles. But two figures are different from the third one: they are right-angled. For this reason they are called rectangles. One of these two rectangles is special: all its sides are

equal. It is called a square. So the square has three names: it can be called a square, a rectangle, and a quadrangle, and each of the names will be correct. My information is accepted, but not without resistance. The children stubbornly prefer to think in terms of non-overlapping classes. Ten minutes ago they argued whether grandpas and dads were men, and men were people. Now they won't call the square a rectangle: either this, or that. I carry out a real campaign for the equality of squares among all the rectangles. By and by my propaganda starts to have an effect. We sum up again: "how many squares do we have? One. Rectangles? Two. Quadrangles? Three." Everything seems all right. And I ask the last question—you remember, the one at the beginning of this paper: "and what is there more in the world, squares or quadrangles?" "Squares!" shout the children in chorus without a shade of doubt. "Because they are easier to cut out", explains Dima. "Because there are lots of them in the house, on the roof, or on the chimney", says Gene. This is the start of the plot. The denouement occurred *a year and a half later*, without any preparation and apropos of nothing. Once in summer walking in the woods, Dima told me: "You remember, Dad, you gave us a problem about squares and quadrangles—what is there more of? Well, I guess we gave you then a wrong answer. Actually there are more quadrangles". And he quite intelligently explained why. Since then my credo is: *Questions are more important than answers.*

Psychologists have carried out and still do numerous experiments trying to teach children some initial mathematical regularities. Consider one of them. First a group of children is tested on whether they understand the following simple regularity: if a piece of clay is kneaded, rolled out, or given another shape, its amount will not change. Those who do not understand this are divided into two groups. One group is left alone: this is the so-called control group. The other one is to learn the law of preservation of the substance quantity: the experimenters show, explain, weigh, and compare. About two weeks later the two groups are tested again to see who has learned anything. More often than not, it turns out that in both groups the progress is quite insignificant and completely the same. Psychologists are usually puzzled: why have the children that were taught so thoroughly not learned anything? However, reading these papers, I asked myself the opposite question: why have the children *who were not taught anything* (the control group) also slightly advanced? After a few years of my lessons with children I may offer my hypothesis: *because they were also asked questions.*



Return to our session. The next problem is another variation to the same theme: preservation of the quantity of objects. The six matches that have remained on the table after the previous task are put in a row. I ask the children to put a button next to each match. Now comes the standard question: "What is there more of: matches or buttons?" "Equal." I sum up: "So there are as many buttons as there are matches". I hide the buttons in my fist and ask the boys to say how many buttons are hidden in my fist. The typical thing is that none of them makes an attempt to count matches. Indeed, what for? I ask them about buttons—so they must count buttons. Dima, as a person most intimate with me, tries to open my fist; others ask in bewilderment: "How can we count them?" I laugh: "Of course, you can't count them, they are hidden, but, perhaps, you can guess somehow?" A tornado of guesses showers down on me; more often than not they have no ground whatsoever. Each boy shouts something else, and only Gene shouts the correct answer. I try to listen to him and to ask why his answer is like that, but he withdraws. Gene is in general too timid. While the boys shout and interrupt one another, he seems to shout the correct answer more often than the others. But when I manage to calm them down and ask him personally, he feels embarrassed and withdraws.

My problem with Andrew is of a different sort. He is a very purposeful boy and obviously lacks motivation at our sessions. When next time I gave them the same problem in another disguise—not buttons and matches, but soldiers with guns, then soldiers left and guns remained, and the scout had to spy out how many soldiers there had been—he was the first to solve the problem. He also likes games where someone wins. But I sometimes lack imagination to present the problem in a suitable form. Besides, the rest of the children do not need it. Dima does not like to solve someone else's problems—he likes to invent problems of his own. It was hard to find an approach to him. Finally I began to say something like, "think of a problem where . . . " and told him my problem. Besides, his solutions are often bizarre (that will be especially evident in the problem to follow); it is hard to hold him within the riverbed of common sense.

There are, of course, problems, with Pete, too; how can I alone manage them all? Goodness me, I have only four pupils and I am unable to provide them with the individual approach. What then can a teacher do, alone against a class of 40 pupils? A teacher is often compared to a conductor of an orchestra. But I seem to myself more like a juggler whose sticks will next moment be scattered on the circus ring. While I try to discuss things with Gene, Dima has already pulled out the cards for the next task, "the odd one out" asking: "Dad, is it the next problem?" The remaining two are snatching the cards out of his hands and, without the slightest respect to the overnight parental labor, crumple them mercilessly. Gene is also looking sideways at them. I open my fist, we quickly check the number of buttons, and pass on to another problem.

The rules of the game, "the odd one out", are well-known. Children are given four pictures answer correctly, but their explanations are far from always being correct. "The suitcase is an odd one!" "Why?" "Because it's not a rabbit, a squirrel, or a hedgehog." "Is that so? To my mind, the rabbit is odd, because it is not a squirrel, a hedgehog, or a suitcase!" The boys look at me in bewilderment and insist: "No, the suitcase is odd!" I want to know whether it is possible to name the three non-odd objects by one common word. At last Pete, who has better mastery of the vocabulary than the others, finds the necessary word: "animals". (He was often very helpful in similar situations.)

By the way, I also give sets with a non-unique solution: (e.g., a sparrow, a bee, a snail, and a plane). It is possible to say that a plane is odd (inanimate), or a snail (cannot fly). At one of the later sessions we had the set of pictures where any of the figures could be odd, depending on the attribute we chose as the basic one (color, shape, size, or presence of a hole) (see Figure 7.). I "nominated" the odd figure, and the boys had to explain my nomination. In this way I tried to convince them that a correct explanation was more important than a correct answer: the prototype of the general mathematical idea of the necessity not only to make correct statements but to *prove* their correctness.

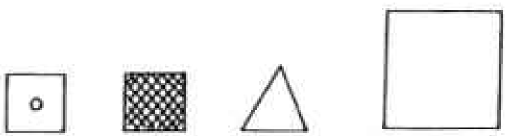


Figure 7.

The pattern, "the odd one out" and its modified versions is very convenient in teaching children to guess the regularities. (This aspect of mathematical thinking is completely forgotten by school mathematics.) Sometimes it is better to take eight pictures that will be divided into two equal groups of four pictures each by the chosen attribute: this pattern was used by M.M. Bongard in his famous book, the *Problem of Recognition*. Quite complicated logical problems arise when we use overlapping classes. For instance, five pictures are to be divided into two equal groups, each containing three pictures; one picture will be common for both groups. An example is: a ball, a tire, rubber boots, a coat, a cap. Three of the objects are made of rubber (a ball, a tire, rubber boots) and three are articles of clothes (rubber boots, a coat, a cap); the common element is rubber boots. A separate problem is how to divide in practice five pictures into two groups with three pictures in each; it certainly won't do to tear the picture into halves. We used the standard technique: two string circles (See Figure 8.)

Dima was a permanent problem. He would say, "Though it's an uncle, he looks like a lady" and would put a picture of an old man with a long beard into the company of women. As to the tire, he tried to persuade us it too was an article ... When nobody agreed, however,

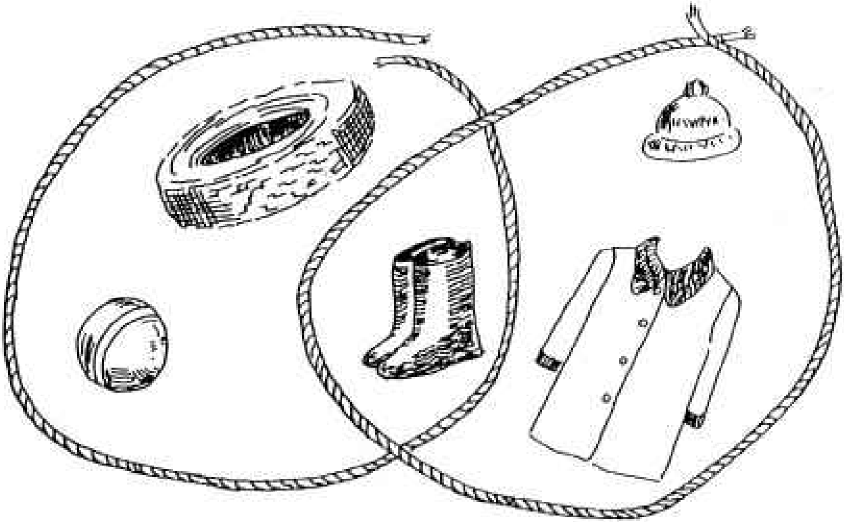


Figure 8.

he declared, "All the same, it's clothes, 'cause cars wear it." Someone will say: this boy can think in a non-standard, creative way. I agree with the first part—his thinking was non-standard; as to creativeness—a really creative person can suggest a sudden, non-standard solution but remain within the boundaries of a problem. Dima so far has only the first constituent, but he is unable to stay within the boundaries of the problem or at least in its vicinity. It is important to develop this faculty without suppressing the other one. But I don't know how to do it. However, despite these doubts, the advantages of this type of problem are undoubtable. There is only one difficulty: their preparation takes too much time. To prepare a problem for 3–5 minutes, I had to spend about two hours overnight. I cannot always afford it. If I were handy with a pencil, I could make sketches directly at the session. But I could never bring myself to do it.

Our next problem (the last one in this session) is geometrical. I get out a box of colored mosaics for children which was bought when I didn't yet think of the mathematical circle, so I—helas!—have only one kit. It is a square field with 15 × 15 holes. These holes can be filled with chips of the one shape and five different colors. The color of the chips is very bright and pleasant to look at. Our problem concerns symmetry. First I lay out the axis: a one-colored vertical line crossing the field in the middle. I call this line, "a looking-glass": in a moment different shapes will "look in the looking-glass". I make various small shapes on one side of the axis, while the boys have to make symmetrical shapes on the other side. I vary everything which can be varied: color, size, location of the shapes. (At the sessions to follow I'll also change the direction of the axis: first it will become horizontal, and then diagonal.) We check our solutions by means of a real looking-glass: is there behind the glass what we see in it? The boys cope

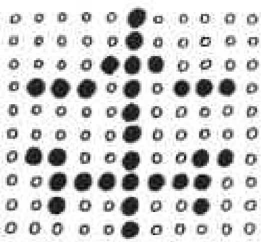


Figure 9.

with the task with an amazing ease, and make almost no mistakes. I can't understand why this subject (the axial symmetry) causes so many troubles in the sixth form (about 12–13 years). Later we devoted many sessions to it. Indeed, symmetry is a rich topic. We examined pictures with symmetrical ornaments in the popular books on mathematics. We drew symmetrical figures with colored felt pens on checked paper; we made symmetrical inkblots folding a sheet of paper in two; we cut out Christmas snowflakes; we found errors in symmetrical drawings (the errors—violations of symmetry—were deliberate); we found among eight pictures four symmetrical figures and four non-symmetrical ones; we found all possible symmetry axes in one figure. Other kinds of geometrical transformations—central symmetry, rotation, translation—prove to be somewhat more complicated for children, but the axial symmetry is a brilliant success.

By the way, the mosaic became my favorite tool. This is not just a plain game, but an actual treasury of all kinds of problems—in geometry, logic, combinatorics and discovering regularities. Once, it gave me an unforgettable lesson as to what is more important for children. It was like this. The boys enjoyed our sessions and often responded to my words, "the lesson is over" with a request to go on. I was naturally proud of myself until I noticed that their requests to go on with the lesson followed exactly when we played with the mosaic. I decided to check my guess. Next session we had no mosaic, and my suspicions were confirmed. I say, "the lesson is over" and the children go away quietly.

I was full of doubts. Indeed, mosaic is very attractive, no wonder the boys are fond of playing with it. But my mathematics (so I thought) has nothing to do with it. I thrust it on them as an unwanted burden, as an unnecessary make-weight to the interesting toy. Next time I staged the crucial test. Again, we worked with mosaic, and again the boys do not wish to finish the lesson. Then I say, "All-right, I have to finish the lesson, but you may play with mosaic." My words are met with a unanimous yell of indignation, and Pete sums up the general viewpoint with the decisive words, "No-o, we want a problem!" That's how I understood what the truth was. Children need intellectual/aesthetical pleasure of full value. If one of the halves is absent, the full value is lost, together with the festive feeling. A Christmas tree without toys is as unattractive for children as toys many years later when my boys are engaged in a more abstract, "intellectual" mathematics they will enjoy it more than other children of their age, because the abstract images and concepts emerging in their brain will somewhere in the depths of their consciousness be emotionally illuminated by the memories of the multi-colored joy of their childhood.

Now each of the boys makes two problems on symmetry. It's time to call it a day, but the boys won't stop. I feel they are tired. And then comes a sudden insight, "Now you give me problems, and I'll solve them". The children are enraptured. With fresh enthusiasm they construct shapes, and I make the symmetrical ones. I do my best. Then another idea occurs to me: I start making deliberate mistakes. Pete is the first to notice it; no end of happiness. The boys seem to have acquired the second breath. They watch my hand with unblinking eyes and greet each mistake with savage war cries.

However, it's time to finish the session. I close the box of mosaic, thank everybody, and declare the lesson to be over. "And when will there be tricks?", remembers Andrew suddenly. "Why, Andrew! It was you who made tricks! You couldn't see the buttons, they were hidden in my fist, but you could count them." As a matter of fact, it wasn't he who counted buttons, but Andrew forgot about it since he looks quite satisfied. We stand up. I cast a glance at my watch: can it be that only 25 minutes have passed? In a few moments the children will go away and I'll stay to put my thoughts in order, to invent new problems, methods, and strategies. And, besides, to cut out, to glue, to paint—in a word, to prepare what in pedagogics is boringly entitled "didactic material" I have only one week until the next session.

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Children and $\binom{5}{2}$

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This is a continuation of the story that I started some time before, about the mathematical circle for pre-school children. This time the mathematics we shall deal with is more like a real one, as the children have become one year older.

In the previous article, I told you about one session of the circle. Today, the story will be somewhat different: I will be telling you about one topic, or, rather, one problem that has been travelling from one session to another undergoing certain transformations. This story will be accompanied by first hardly noticeable and then more insistent emergence of the central idea of mathematics, that of the proof.

When I decided to present my notes to a broad audience, the thing I was most apprehensive about was that someone would surely take me for another prophet proposing just another method of cultivating infant-prodigies. The topics of our sessions sometimes do sound in a depressively scientific way: probability theory, programming, topology, combinatorics, etc. I can well imagine an enthusiastic and easily carried away reader educated by lectures like *The Unknown Potentials of Our Psychic Faculties* who would exclaim, "Only think, he teaches probability theory to toddlers! His little ones are experts in the subject in which college graduates understand nothing at all!" But I may as well imagine another character, more skeptical and reasonable, who would grumble, "It's beyond me why we should stuff children's heads with all that nonsense. Let them have normal childhood."

I personally would not be happy to hear either of those remarks, since both viewpoints are based on misunderstanding. We never 'studied' any formulas or theorems of mathematical probability theory. I do not believe there exist children, no matter how gifted they are, who would be able to do that at this age. What should we do instead? As the first step, it is advisable to ask oneself: whence does probability theory come from? What are its roots? It is quite obvious that, like so many sciences, like arithmetic, e.g., probability theory originated as a result of observations over the phenomena of the real world, namely, over random, unpredictable events. The next step is to understand that similar observation can be made together with children. Not all of them, of course, only the simplest ones. Besides, you probably noticed that children make

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such observations on their own, for instance when they play a game with dice (scored from 1 to 6). What remains to do is to just slightly emphasize the probabilistic nature of such observations.

In what way? Actually, there are many ways to do that. For example, you may give children, instead of a regular die, a crooked polyhedron, so that they would notice the game became 'unfair': some digits are cast more often than others. Or else, you could invent a game in which they would have to count the sum of digits on two dice. They are sure to notice sooner or later that, e.g., the sum 7 occurs more often than 2. Or you may invent a toy thermometer that instead of temperature would measure probability: you would bring it close to a number and children, by pulling a cardboard tag, will tell you whether it is 'hot' or 'cold', i.e., whether this number occurs often or seldom. In this sort of activity we are limited only by our own imagination and the potential of real children. If the children have grasped something, if something has stuck in their heads, it is fine. If not, then we merely played together (and you know that children always enjoy playing with grown-ups).

Let us sum up: what we shall try to teach the children will be not science as it is, as a ready-made product of the previous generations, but the preliminary observations that gave an impetus to its formation. I would like to examine one example in greater detail. It is a simple problem, but it gives rise to many reflections. It involves psychology, pedagogics, mathematics (and even a bit of philosophy): all of them are tied together in a knot. Well, here goes.

The problem itself belongs to combinatorics. In Russia this branch of math used to be studied in the 9th grade (15–16 years). Then, it was considered too difficult (recall what a fright was the binomial formula) and excluded from the curriculum. Actually, all the troubles of the students stemmed from the fact that they had to begin with formulas, while they did not have the subject at their finger-tips. The latter metaphor is to be understood in its literal sense. Combinatorics deals with counting the number of different combinations of objects. But the objects themselves are not there, they have to be imagined, as well as their combinations. If only students could begin by combining real dice, chips, etc.

The session begins. We sit around our mosaic kit. The task is to make 'beads': a chain of five chips, two of which must be red, and the remaining three white. This can certainly be done in various ways. Our task is to find all the ways to do it without repetition. Scientifically, these sequences are called combinations of two elements out of five; their number is denoted $\binom{5}{2}$ and $\binom{5}{2} = \frac{5 \times 4}{2} = 10$. The children certainly do not know all that and will not learn it in the course of our sessions. They merely make beads in turn, one after another. Each result is checked by the whole company to see whether it is really a new one or coincides with any one made earlier. Sometimes we have a dispute, for instance (Figure 2) whether it is the same solution or a different one. I explain that they are different.

The principal problem of combinatorics is how many solutions there are. But

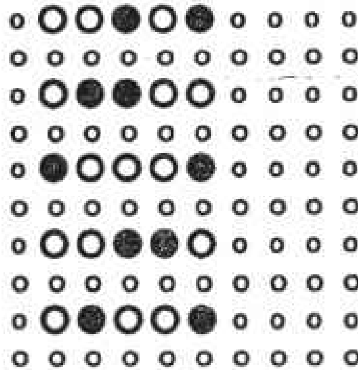

Figure 1.

the boys are still very far from it. They do not see any obstacles to playing this game infinitely, but, on the other hand, they are always ready to agree "there are no more new beads". They do not in general see as yet any difference between "it is impossible" and "I can't do any more", and they express firm certitude that I, being adult, will by all means be able to make the 11th solution, the 12th one, as many as I would wish to. I have to undertake the task myself. Contrary to the boys who enumerated their solutions helter-skelter, without any system, I demonstrate the ideally systematic approach and sort out the solutions in a strict order: first, I put one red chip in the first position, while the second red chip is placed in turn in the 2nd, 3rd, 4th, and 5th position. When this sequence is over, I put the first red chip in the 2nd position, etc. Do you think the boys are impressed? Not in the least. The only thing they understand is that I also failed to think of the 11th solution. They can already tell one solution from another, but it is yet beyond their capacity to distinguish between order and disorder. This problem has to be postponed for at least half a year. (And meanwhile it would perhaps be helpful to teach them to put their toys in their places. I wonder whether a tidy room is in any way connected with tidiness of reasoning).

Half a year has elapsed. It would be silly, however, to present the same problem to the children. I have got another idea: preserve the mathematical essence of the problem but change its external, physical formulation. This time each of the boys gets a sheet of paper with several sets of five circles linked by lines. Two of these circles are to be colored, and the remaining three left uncolored.

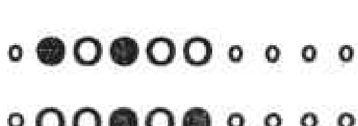

Figure 2.

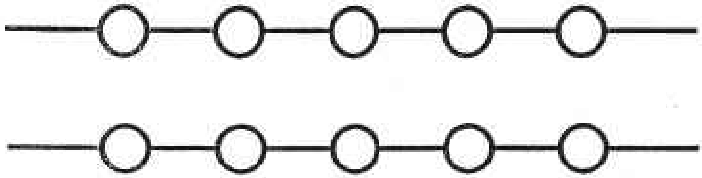

Figure 3.

It goes without saying that the solutions must not be repeated. The boy who will find the greatest number of solutions will be the champion. (v. Figure 3).

And there is also a trick that could seem insignificant, but it is not: I hand out pencils of different colors and take pains to ignore this fact in further discussion, because the solution is different not when the beads have been colored with another pencil but when another pair of beads has been selected. I hope that will emphasize for the boys the purely combinatorial nature of the problem. (Later, in another group of children I made beads round, square, and triangular.)

A few minutes of independent work (which shows, by the way, that a problem to be solved on paper is much more difficult than on a mosaic field) are followed by a noisy exchange of opinions and results. This time each boy has obtained 10 solutions. "Do you remember that we have once had a similar problem?" It turns out that I am wide of the mark, because I substituted my own viewpoint for the boys' feeling. What exactly does 'a similar problem' mean? To me, it is quite obvious that a similar problem is the one that also deals with a combination of two elements out of five. For the boys, however, a similar problem is such that demanded drawing with colored pencils. I am not for prompting, but this time I simply have to. The boys snatch the mosaic kit happily, make beads, and it even occurs to them to compare their mosaic solutions and those on sheets of paper. Somebody recalls that last time we also had 10 solutions. That at last makes them for the first time express their doubts: is it truly impossible to make more than 10? I smile mysteriously and pass on to another assignment.

The problem proved to be a gold-mine. Soon it appeared for the third time, the fourth and even the fifth one, though somewhat transformed by the new clothes.

Each boy receives a sheet of checked paper with a 3×4 rectangle drawn on

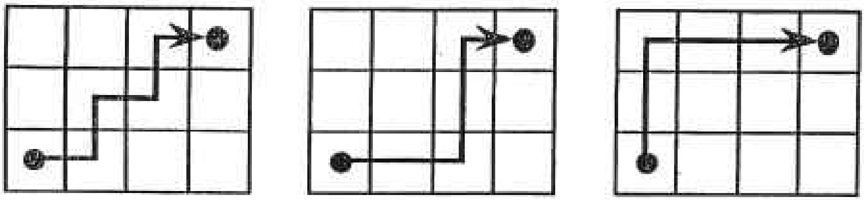

Figure 4.

it. (After a brief dispute whether it is a square or not, I can formulate the problem). The task is to draw a path from the left lower corner into the right upper one, under one constraint: each time it is allowed to move either one step up or one step to the right. As before, one has to find all the versions without repetition. It may not immediately leap to the eye what this problem has to do with the previous one. It will become clear in a minute.

Work is in full swing, and the increased skill of my pupils is quite evident: they make less mistakes and find all the 10 solutions rather quickly. (I already foresee another trap: the boys may start to think that the answer to all combinatorial problems is 10. I will have to give them more problems, the answers to which will be different). Now, we must discuss the most important question: how many steps must we make to the right and how many up, to go from one corner into the other? But first we have to agree what a step is: my opinion is a step is a passage from one square to the adjoining one, while the boys believe any straight segment to be a step. Finally, the agreement is achieved. The answer seems to be obvious. But no, again. I am in total bewilderment and long after the session is over try to find the reason for that. Indeed, the question seemed simple only due to my stupidity: I overlooked the fact that the property of the number of horizontal and vertical steps being equal for all the paths forms the basis for the coordinate representation of vectors, namely, that when vectors are summed, their coordinates are also summed. I remember fairly well my own amazement, already as an adult, at this property of vectors. This material may serve as a source for a series of problems, it may even help to hint at negative numbers (if steps backward are allowed, but registered with the minus sign).

Meanwhile, we carefully count the steps: each path proves to have exactly three steps to the right and two steps up. Therefore, at our next session, we put down the following sequences: UURRR, URURR, etc., each containing three letters R (denoting a step to the right) and two letters U (for the steps up) (v. Figure 5).

You should have seen how excited the boys were when I showed them this connection. They demanded that I cut the sheet with the letter combinations, and pushing one another aside, started to attach each clipping to the corresponding path. I remained a detached on-looker but nevertheless made an attempt to

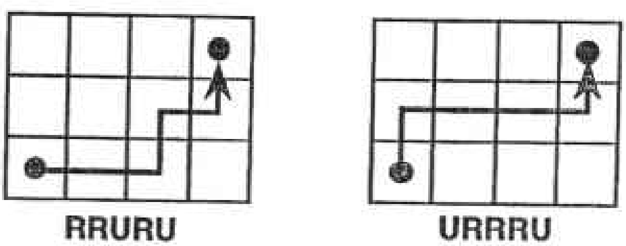

Figure 5.

suggest an idea, as if by chance, "Perhaps, we will be able to find some more solutions, the eleventh one, the twelfth one—" The only one to respond is Gene, "No, says he, we have 10 solutions here and 10 solutions there." "But they can be different? 10 solutions here and other 10 solution there?" By this moment, however, all the clippings are distributed, and my suggestion has not been justified: both groups of 10 solutions fully coincide, or, in terms of mathematics, there is a bijection between them. It was nevertheless important to put the result to doubt, if at least for a moment, only to be able to appreciate it as a true result several minutes later.

Now, on the wave of enthusiasm, we can move a bit further. "Tell me, boys, is it possible to denote the steps up and to the right by any other letters? Not by U and R, but by other ones?" — "Certainly, by any letters!" — "By which, for example?" — "For example, by A and B", says Pete. "Or, for example, by Z and Q", this is Dima's suggestion. "Or, the step to the right by + and the step up by a comma" (this is my idea). "O-o-o", the boys laugh happily. "Or, I continue impassionately, "denote the step to the right by a red circle, and a step up by a white one". — "How?" — "Like this". I take one of the drawings (e.g., this one, Figure 6) and the corresponding clipping and draw near the following picture.

In the ensuing pause, the pause before the explosion, I still have time to link the circles with lines, after which the drawing becomes exactly the same as in the second problem. They have recognized it, no doubt. Each new insight is accompanied by wild joyous shouts and prancing. The clippings and drawings on the table are in a mess, and it is utterly impossible to go on. It is time to finish the session. Now I can make a pause for a month and distract my students with other problems. Let the idea settle in their heads and take roots. Besides, children may get tired of similar problems.

The next episode concerns the final stage. I have put on the table 5 boxes and 2 balls; the task is to put the balls in the boxes in various ways (the remaining three boxes must stay empty). At the beginning the work goes on briskly, but approximately at the fourth step the heated discussion breaks out: have they already got this solution or not? The boys ask me to be the judge but I pretend not to remember either. What is to be done?

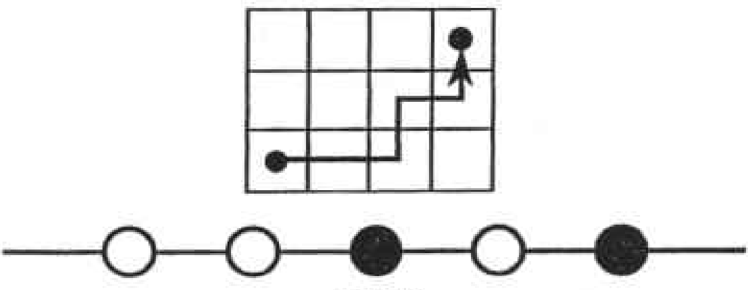

Figure 6.

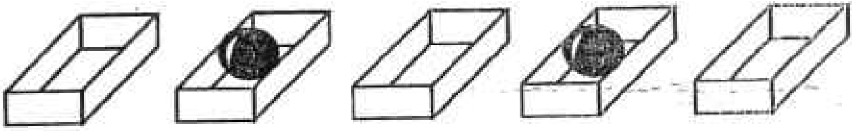

Figure 7.

By the way, not every child will understand what should be done in a situation like that. One has to *denote* an empty box by some symbol and a box containing a ball by another symbol, and record all the solutions. But the unassuming word "denote" is underlined by the immense idea born and evolving side by side with human civilization. It will suffice to recall the as yet enigmatic history of the origin of writing, the evolution of pictograms into hieroglyphs, and hieroglyphs into alphabetical systems, etc. Throughout its long history mathematics has always been engaged in inventing and improving systems of notation, first for numbers, then for algebraic operations, then for more and more abstract entities. As late as in our century the studies of symbolic systems turned into an independent science, semiotics. (It is not by chance that 6- and 7-year-old first-graders are so puzzled when the teacher tells them, "Denote a syllable by a rectangle, denote a vowel by a red circle, a consonant by a black circle; denote the unknown number by letter X . . ." It is so simple and obvious—for you and me; *denote*, and that is that. But children are stupefied.)

What I always tried to do in our circle was not only solving independent problems, but also formulating (at least, for myself) certain super-goals. One of them consists in acquainting the children with the ideas of semiotics. More than once did we discuss that numbers are denoted by figures, speech sounds by letters, and musical sounds by notes. We recalled other symbolic systems, like traffic signs. And whenever it was possible (and useful) we invented symbols for various objects we manipulated with. Therefore, the idea of denotation was not entirely new to the boys. This is why they suggest that we 'draw' the solutions. At first, they actually try to make quite true-to-life sketches; I would say they are on the pictographic level. This is, however, rather difficult, so they soon pass on to hieroglyphic drawings; the sketches become more abstract: an empty box becomes a square, and a box with a ball—a square with an inscribed circle. I suggest that in the latter case they simply draw circles. Another trouble is that the children cannot yet draw accurately and it is not always easy to distinguish between their circles and squares. I make another suggestion: when you draw a circle, make a cross inside. The result is as follows:

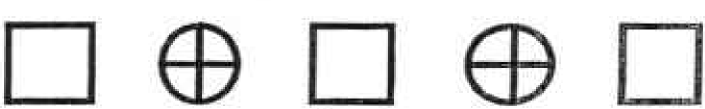

Figure 8.

"Why with a cross?" ask the boys. "What difference does it make how to denote?" answer I trying, by indifferent shrugging of the shoulders, to hint at the arbitrary nature (within certain boundaries) and relative independence of the sign with respect to the object denoted.

By the way, the problem we are considering is in one respect more complicated than those we had before, because each solution has to be compared not with the preceding ones, but with their *notation*. This time the boys manage to find only 9 solutions and after several failures come to the conclusion that no more solutions can be found.

Lo! Now comes the moment of my triumph, the moment that I have been preparing and anticipating for such a long time. Suddenly Pete exclaims poking his finger in the sheet of paper, "Hey, look, R, R, U, R, U!" Dima also jumps up, very excited, "Yes, Dad, I wanted to tell you long ago!" "That means there is one more solution," catches up Gene. Dima suggests, "Let's bring the solutions of that old problem and see which one is missing".

Of course, we do not have to go too far: quite 'by chance' the envelope with the solutions of all the preceding problems happens to be on the table. Which problem should we take as a basic one? The boys want to take the clippings with beads, and very soon, on the fourth step, we find the missing 10th solution. (No triumph is free from a little embarrassment: when we put out the clippings with beads, one of them turned upside down. As a result, one of the solutions was lost, and the symmetrical one occurred twice. We were almost completely confused).

I have a feeling that what has happened today is very significant. Not only have we solved the problem, but we have done that by *reducing it to another, isomorphic one*. This represents an extremely important general idea of mathematics. I also find it wonderful there exists material that gave me a chance to demonstrate this idea to six-year-olds, and in a manner that enabled them to arrive to it themselves.

* * *

As a rule, events at our sessions change one another at a neck-breaking speed. We have not yet finished up with one great idea, as another one is approaching. Quite by itself the question emerges why we get 10 solutions each time. Is it true that there is no more, or we simply failed to discover other ones? How can we *prove* there are only 10?

The proof. The central notion of mathematics. I would even say, a formative notion placing mathematics apart from all other sciences. The concept of what is a proof and what is not, changed throughout the centuries and acquired its modern form only at the beginning of the XXth century (this very interesting story can be found in the book by Morris Kline *Mathematics. The Loss of Certainty*). Mathematicians of the previous centuries used to regard as convincing the type of reasoning that in our days would be indignantly rejected by any schoolteacher. As a matter of fact, we are dealing here with a very odd phenomenon: by force of abstract reasoning certain assertions become more convincing than other ones. A very intelligent high school student once asked his teacher, "The fact that the angles at the base of an isosceles triangle are equal is completely obvious, you can see it in many examples. Nevertheless, this ought to be proved. On the other hand, the fact that voltage is equal to strength of current power multiplied by resistance is in no way obvious. This fact, however, is not proved but demonstrated by several examples." This question is an extremely rare attempt to penetrate into the core of phenomena. I am sure that the majority of schoolchildren regard the proof as a kind of mathematical ritual. This is the way to do it, and that's that. I cannot help remembering the story dating back, as it seems, to the XYIIIth century, about a student who said to his teacher, "Is there any need in these ambiguous speculations? We are both gentlemen, so your word of honour that the theorem is true will be enough."

Funny, isn't it? But we modern educated people, are we not the same? We read in books that the age of the Earth is 4.6 billion years and how many of us ask ourselves, "How do they know?" Have you ever found in history manuals any proofs that the events occurred exactly there, then, and in the way they are described (if they took place at all)? You will never find a trace of proof, but, strange as it may seem, that does not in the least decrease our faith in the facts exposed in the book. "The gentleman's word of honour" (the gentleman being the author of the book) proves to be quite convincing. As you see, the problem is not that simple, even when it concerns adults.

But what does it have to do with children? The point is that it is necessary to grasp the problem of the proof as a whole, only then shall we hope to succeed in discovering some clues, some points of contact between children's mentality and the idea of the proof. Throughout my years of work with children I kept looking for such points of contact wherever possible. The first attempts were represented by the game "Which one does not belong?" with *non-unique answers*. In playing this game I drew the boys' attention to the importance of not only correct answers but of the correct explanations. Then other problems involving the necessity of proof started to appear: prove that we see with our eyes and hear with our ears, not vice-versa (the proof: if you close your eyes, you can't see any longer; if you close your ears, you can't hear); prove that clouds are closer to us than the sun (the proof: clouds cover the sun); prove that we think with our head, not stomach (I have so far failed to come out with a good proof). Sometimes I used the following method: I made wrong statements on purpose, provoking the boys to prove their falsity to me.

What could be an analogue of the proof in our combinatorial problem? It seems that it can only.be systematic enumeration of all possible combinations, to make sure we have not missed any of them. Half a year ago the boys did not take to the idea. But perhaps they have matured to it by now? Let us return to the discussion that we have started to describe but dropped at mid-sentence. How can we be certain that there are no solutions except those 10 we have found?

Dima is of the opinion that we should try for many years and if we still find nothing, that would mean there are no more solutions. I bring forward a natural objection: and if there are? Gene is quite pessimistic: he declares he is certain *he* won't be able to find any more solutions. Pete asks me: is it true that I myself do not know how many solutions there are, or I know it and ask only for fun. I confess that I know exactly how many solutions there are. After that the boys cease to understand altogether what it is that I want. Then Dima suddenly utters a rather incoherent phrase which, however, contains the words "the leftest box." I snatch at it, quickly interpreting it in the necessary direction, and begin to reason outloud.

Indeed, let us take the first ball and put it into the first box from the left. Where can we now put the second ball? Into the second, third, fourth, and fifth box, which gives four solutions. Now put the first ball into the second box. Where can we now put the second ball? Into the first, third, fourth, and fifth box, which yields four more solutions. Thus we have all in all 4 solutions 5 times, i.e., 20 solutions!! Can you beat that?! The boys are completely stupefied, and I finish up the session as quickly as possible.

This time I hit the mark. Now all the members of our circle without exception carried out an independent research of their own: they manipulated with things at home, drew something and in the long run, some sooner, others later, sometimes with my help, managed to understand why to obtain the correct answer 20 must be divided by 2.

* * *

At some later moment, not at the circle session but obviously under its effect I had a noteworthy discussion with my son which I am going to relate here. Dima asked me how it is in general possible to convince other people. "There are many ways to do that," I said. "For instance, in physics they carry out experiments". "I see." (Dima was already acquainted with the notion of a physical experiment from the book by L. L. Sikoruk, *Physics for Little Ones*, one of the most brilliant masterpieces among popular scientific books for children). "Can you tell me, for example, which things fall more quickly: heavy or light?" "Of course, heavy things fall down more quickly." "Well, that's what *you* think. And another person may say that all things fall down equally quickly." "N-no, they won't." "Why not?" "If you take a stone and a sheet of paper, the stone will fall down more quickly." "That's right. So, to convince this person that he or she is wrong, you will make an experiment, won't you? You will take a stone and a sheet of paper and see which one will fall down more quickly." "Yes". "And now let us make another experiment."

The idea of the experiment to follow was prompted to me by a friend. First we take two similar sheets of paper and they naturally fall down equally slowly. Then I crumple one sheet into a little ball. I am going to ask which one will fall more quickly this time, but Dima forestalls me, "And now this one (he points to the crumpled sheet) has become heavier!" — "Why?" — "Because it will fall down more quickly."

So this is how it is. To be convinced by a physical experiment, one has to have sufficiently mature logic to understand the inadmissability of logical circles.

There is no stopping me, however. We keep dropping pairs of objects that are at hand: a button and a big heavy sheet of Whatman paper; a button and a weight; a plastic block empty inside and a wooden block of the same size, etc. Dima is puzzled by the results: he was going to predict that a button would be heavier than a Whatman sheet, but quickly gave up the idea. "So it happens in different ways: sometimes it is light things that fall down more quickly, and sometimes it is heavy ones." He is almost ready to be satisfied with this allegedly correct explanation. But suddenly understanding comes, "Aha, dad, now I see! It's the air that does not let them fall." "Let whom?" "The sheet is big, and the air does not let it fall, it interferes. And the button is small, and the air interferes less." "Right! And if there is no air, what will happen?" "Then all things would fall equally." - "Good for you. And what happened when I crumpled the sheet of paper?" While Dima tries to find the words to answer me, my impatience lets me down and I answer instead of him, "The air does not interfere any longer." Dima corrects me, "Yes, it does, but it interferes less."

In my previous paper I stated my educational principle: never try to impose your view on a child, even by a hint. But there is a more important principle in the hierarchy of principles: never follow any principle in an inexorable way. I feel now it will be pertinent to drop the former principle for a while. With an obvious hint in my voice I ask one more question concerning the crumpled sheet, "Do you still think it actually becomes heavier?" Dima laughs in reply. "No, of course, not!," says he, "Perhaps, only just a little bit heavier."

In the evening, recording our conversation, I think it over more thoroughly. Suddenly I notice that what we made was not exactly a physical experiment. An experiment is a question posed to nature when the answer is yet unknown. In our case Dima knew the answer beforehand. It was not necessary to drop a weight and a button: the child's own experience in the physical world would have been enough to predict correctly the results of the experiment. We may well say that none of the above experiments gave him any new information, if we mean facts. Actually, we have achieved the proof by enumerating all the logical possibilities, like in the problem with balls and boxes. That partially explains why *questions* are so useful. By means of questions we help the child to compare the elements of experience which used to exist separately and were not related to one another.

* * *

To finish with the idea of the proof, I would like to retell to the readers another episode that occurred when we were staying in the country in summer. Pete and Dima remembered their recent visit to the Zoo where they were shown monkeys. Suddenly I interrupted their talk to declare it was not them who had been shown monkeys, on the contrary, they had been taken to the Zoo to be shown *to* monkeys. A heated discussion followed. The first argument, "We looked at them" I easily counteropposed, "They also looked at you." The second one was more serious, "We can go wherever we like, and monkeys stay in the cages." But I coped, "No, you can't go wherever you like. The monkeys are not allowed to leave the cages, and you are not allowed to enter them. There is a grating, and the monkeys go wherever they like on one side, and you on the other side." The dispute went on for some time, and suddenly Dima exclaimed happily, as if he had caught me, "Hey, Dad, this is again mathematics!"

At the very first session of our circle the first thing the children did was to rush to count buttons on the table. This is what mathematics was for them: it's when they count. Since then the idea of what mathematics is has undergone a considerable change in their heads. I would allow myself to assert that we have come somewhat closer to the mathematics of Lewis Carroll. Carroll's mathematical papers, as well as his tales and rhymes display a striking unity and harmony and are penetrated by miraculous logical play. And I feel that what children need is this mathematics à la Lewis Carroll.

Translated by A.V. Yarkho