Signal Processing 5 (1983) 201-220
North-Holland Publishing Company

ARE IMPOSSIBLE FIGURES POSSIBLE?

Zenon KULPA
Iconics Laboratory, Institute of Biocybernetics and Biomedical Engineering, Warsaw, Poland

Received 24 June 1982
Revised 11 November 1982

Abstract. In the paper, a thorough analysis of the so-called impossible figures phenomenon is attempted. The notion of an impossible figure and some other related phenomena (e.g. 'likely' and 'unlikely' figures) are precisely defined and analyzed. It is shown that all these figures, being illusions of spatial interpretation of pictures, are more relevant to psychology of vision (and related artificial intelligence research) than to geometry or mathematics in general. It suggests an inadequacy of several previous formal approaches to explain these phenomena and to deal with them in computer vision programs.

The analysis of these spatial interpretation illusions allows us to formulate several properties of the structure of our spatial interpretation mechanism. A two-stage structure of this mechanism, a set of basic 'interpretation assumptions' and a set of basic 'impossibility causes' are identified as a result.

Zusammenfassung. In diesem Beitrag wird versucht, das Phänomen der sogenannten 'unmöglichen Figuren' gründlich zu analysieren. Der Begriff der 'unmöglichen Figur' sowie einige verwandte Phänomene wie 'wahrscheinliche' oder 'unwahrscheinliche' Figuren werden genau definiert und beschrieben. Alle diese Phänomene ergeben sich aus dem Mechanismus der räumlichen Interpretation ebener Bilder bei der optischen Wahrnehmung durch den Menschen. Wie hier gezeigt wird, kommt ihnen daher für das Gebiet der Psychologie der optischen Wahrenehmung (ebenso wie für Forschungsaufgaben auf dem Gebiet der künstlichen Intelligenz, soweit sie damit zu tun haben) eine größere Bedeutung zu als für die Geometrie oder Mathematik allgemein. Manche früheren Modelle, mit denen versucht wurde, diese Phänomene formal zu beschreiben und mit Hilfe von Rechenprogrammen quantitativ zu erfassen, können aufgrund der neuen Ergebnisse als unzureichend angesehen werden.

Die Analyse des Mechanismus der räumlichen Interpretation bei der optischen Wahrnehmung ermöglicht es, einige strukturelle Eigenschaften des räumlichen Interpretationssinns zu beschreiben. Als Ergebnis der Studie erhält man ein zweistufiges Modell dieses Mechanismus, einige 'Grundvoraussetzungen' des räumlichen Interpretationsvermögens sowie einige 'Grundursachen' für 'unmögliche' Figuren.

Résumé. Dans cet article, une analyse complète du phénomène des figures impossibles est tentée. La notion d'une figure impossible et quelques autres phénomènes (c'est-à-dire les figures possibles et non vraisemblables) sont définis avec précision et analysés. Il est montré que toutes ces figures, qui sont des illusions d'interprétation spatiale des images, sont plus significatives pour la psychologie de la vision et pour la recherche en intelligence artificielle que lui est liée que pour la géométrie ou les mathématiques en général. Ceci suggère une inadéquation de plusieurs approches formelles précédentes pour expliquer ces phénomènes et pour les traiter sur ordinateur avec des programmes de vision.

L'analyse de ces illusions d'interprétation spatiale nous permet de formuler plusieurs propriétés de la structure de notre mécanisme d'interprétation spatiale. Une structure à deux niveaux de ce mécanisme, un ensemble d'hypothèses d'interprétation de base et un ensemble des causes d'impossiblité de base sont identifiés comme résultats.

Keywords. Impossible figures, visual illusions, spatial (3-D) interpretation of pictures, computer vision.

1. Introduction

Since publication of the paper by L.S. Penrose and R. Penrose [1], the so-called 'impossible figures' or 'nonexisting objects' have drawn some attention of both artificial intelligence (e.g. Huffman [2]) and psychology of vision (e.g. Gregory [6], Cowan [8, 9], Young and Deręgowski [10]) researchers. For the psychology they are interesting as new types of illusion, being a source of additional informations about our spatial interpretation mechanisms. For similar reasons they are of interest to artificial intelligence research, providing cues for organizing algorithms modelling human abilities to see the three-dimensional world in flat pictures. It should be mentioned also that these effects are of growing interest to the theory and practice of visual arts (Ernst [26], Raushenbakh [28], Kulpa [33]). Most of previous works on this subject treated it either in a descriptive and rather loose way (Gregory [6]) or from a geometric and strictly mathematical point of view (Huffman [2], Cowan [8, 9]).

Beside showing several examples of impossible figures, Penroses [1] showed a photograph of a *possible realization* of their 'impossible staircase'. Gregory [6] desdribed also an important example of possible realization of the impossible 'Penrose's triangle'[1] (Fig. 14(c)). He also stressed that such erroneous perception cases are important sources of information about mechanisms of selection of interpretation hypotheses in human brain.

Huffman [2, 3] and partially Clowes [4, 5] investigated the subject in the context of automatic computer analysis of polyhedral scenes. This subject was investigated within the area of computer (robot) vision and artificial intelligence starting roughly from works of Roberts [11] and Guzman [12], extended further by Waltz [13] by taking into account presence of shadows in the scene. Huffman's and Clowes' works concentrated mainly on investigation and compilation of lists of possible (and impossible) corner configurations of edges in polyhedral scenes. But Huffman [2] found examples of figures nondecidable, as regards their impossibility, on the basis of these local corner configurations alone, as well as he noted that impossibility determination depends on the geometrical class of objects the given one is assumed to belong to. A figure, being impossible (i.e. non-realizable) as a three-dimensional object of a given class can be realizable as an object of another class. He also extended his impossibility criteria to make them more global (a so-called 'gain' concept [2]) as well as he devised criteria for not yet investigated class of objects (a so-called 'smooth' objects [2]). What is also very important, he noted the existence of so-called 'unlikely' figures, looking impossible, although being easily realizable, as well as he found figures which, although impossible, do not look impossible at all (see Section 3). Undoubtedly, the Huffman paper [2] has been the most comprehensive and important work to date on the subject of impossible figures.

Mackworth [14, 15] developed a geometrical formalism of gradient space (based on Huffman's dual-space concept [2]). This approach was aimed as a general mechanism of spatial interpretation of drawings, and it was thought to be able to resolve the impossible figures case as well. Kanade [17, 18] relaxed then some restrictions of Mackworth's formalism and generalized it onto wider class of objects (a so-called 'Origami world'). He was then stuck by the great numbers of different interpretations these approaches usually produce (even for quite simple figures), most of which, despite their geometrical correctness, appear unnatural and hardly imaginable to humans. The aim of modelling the human mode of spatial interpretation by means of such geometrical formalism was therefore shown to be bound to failure – the notion of 'naturalness' being hardly expressible geometrically. Finally it turned out that these approaches are unable to distinguish properly impossible figures from possible ones (see e.g. Draper [19]). From a detailed analysis of the gradient space approach and its possible extensions, undertaken by Draper [19], followed that it is generally inadequate as a mechanism of proper spatial interpretation of drawings. It finally led to informal formulation of the so-called 'sidedness reasoning' proposal (Draper [19]), in many respects similar to the conclusions that can be drawn from the analysis of the impossible figures phenomenon only, as explicated in the present paper (Section 5).

Cowan [8, 9] concentrated on formal mathematical analysis of the comparatively restricted

[1] It should be rather renamed 'Reutersvärd's triangle', after a Swedish artist who invented it as early as in 1934, in his works 'Opus I' and 'Opus II' [32, 33].

class of figures, the so-called 'cornered toruses' with square cross-sections and, more particularly, he analyzed in details the four-corner toruses. His main aims were to devise an algorithm generating all such toruses and to classify them, according to their possible/impossible distinction as well as other properties of their internal structure. However, he has made also (although vaguely) an important general observation that quite often some impossible figures look more realistically three-dimensional than some possible ones.

Using the results surveyed briefly above as a starting point, a thorough analysis of the impossible figures phenomenon and other related phenomena is attempted. In Sections 2 and 3, the notions of *impossible figures* and so-called *likely* and *unlikely figures* are precisely defined and analyzed. It is shown that all these figures, being illusions of spatial interpretation (situated somewhere between low-level 'optical' illusions and higher-level 'semantical' illusions, e.g. like that occurring in the Rorschach test), are more relevant to psychology of vision (and related artificial intelligence research) than to geometry or mathematics in general.

It is made clear that only the spatial interpretation, *not* the figure itself, can be reasonably called impossible. Then it follows (Section 4) that eventually *all* impossible figures do have possible spatial interpretations, so that the actual question with impossible figures is: why our interpretation mechanisms do not find those possible interpretations for certain, otherwise rather simple drawings. This question is also closely related to the problem of modelling the notion of 'naturalness' of an interpretation. Therefore, on the one hand impossibility effects can be fully explained only when we have learned the rules underlying our spatial interpretation mechanisms, and on the other hand they provide a rich source of informations about inner workings of these interpretation mechanisms.

The analysis of the impossible figures illusion allows us actually to formulate several properties of our spatial interpretation procedures. In Section 5 we explain a two-stage structure of these procedures, we list several basic 'interpretation assumptions' (which filter out the whole bulk of geometrically feasible, although 'unnatural' interpretations), and we identify a set of basic 'impossibility causes', i.e. these important local features whose inconformities detected at the verification stage produce the majority of impossibility effects.

Main theses of this paper were formulated in a report finished in 1980 (Kulpa [16]). This paper is a revised version of that report, the revision being influenced mostly by further developments of computer methods of spatial interpretation of images, especially the works of Kanade [18] and Draper [19], addressing several of the problems related to the contents of this paper.

2. What are impossible figures?

The following definition summarizes what is generally assumed when the notion of impossible figures is considered:

Definition 1. An *impossible figure* is a drawing making an *impression* of some three-dimensional object, although the object suggested by this three-dimensional *interpretation* of the drawing cannot exist, i.e. an attempt to its construction leads to geometrical contradiction (see also Section 3).

Such figures are sometimes called 'impossible objects' or 'nonexisting objects', as depicting objects that cannot exist (see e.g. Kulpa [20]). Here we shall accept the term 'impossible figures', as more widely used, reserving the term 'impossible objects' to denote real three-dimensional objects having some flat projections identical to some impossible figure (following Gregory [6]).

Now let us take notice of the two words essential for the definition, although usually overlooked by more mathematically-oriented researchers. The words are: '*impression*' and '*interpretation*'.

Firstly a figure, to be judged impossible or not, should make an impression of some three-dimensional object. Therefore, any drawing not representing some such object cannot be judged in this terms at all (Fig. 1(a), (b)). There must occur some structure in the drawing allowing its interpretation as a projection of some three-dimensional object. It makes no sense to speak about impossibility of figures like these shown in Fig. 1(a) or 1(b); they do not display any consistent set of cues allowing to interpret them as projections of some complete objects in three-space.

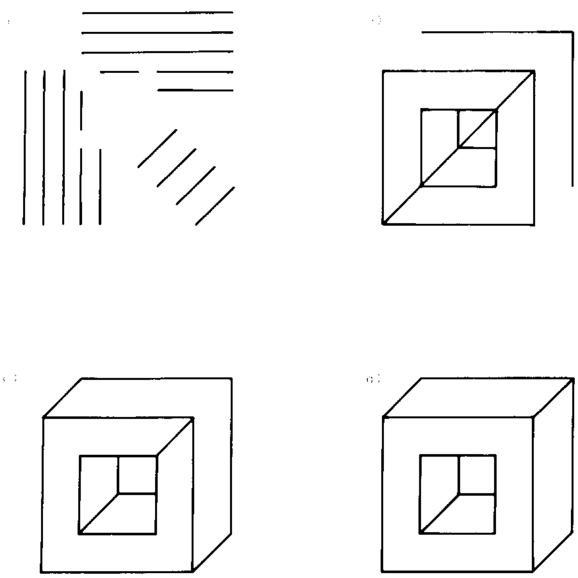

Fig. 1. The same set of sixteen line segments (a) arranged differently to represent non-objects (a, b), and impossible (c) and possible (d) figures.

Secondly, what is even more important here, the word 'impression' is a psychological, not mathematical term. It implies a necessity of human judgement to decide whether a given drawing can be classified as impossible. These subjective decisions are often imprecise and variable with changes in a situation, e.g. a context or the subject attitude to the task (see also discussions of Figs. 4(b) and 5(c) later on). Therefore, the notions of impossibility or three-dimensionality of a flat figure cannot be precisely and mathematically defined – in the last resort always a psychological experiment will be decisive.

Moreover, the strength of impossibility impression can vary for structurally identical figures, but, e.g., of different sizes or proportions. Usually this variation of 'degree of impossibility' goes in accordance with variation of 'degree of three-dimensionality'. It was noted by Cowan [8, 9]; Fig. 2 shows quite another example. In the 'thick' square frame (Fig. 2(a)), the contradictory depth organization cues are too near and they interfere so strongly with each other in the process of three-dimensional reconstruction of the drawing that this process becomes almost destroyed. As a result, the three-dimensionality impression is weak and so is the impossibility impression. The

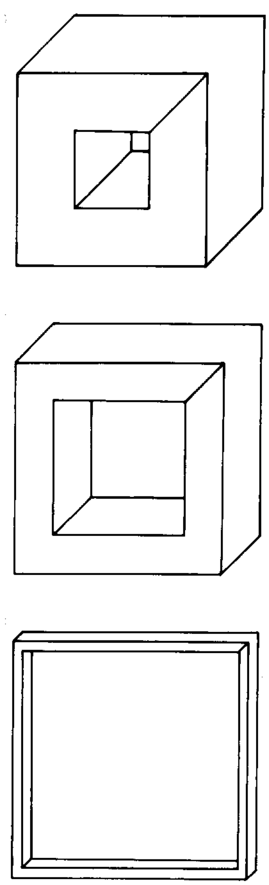

Fig. 2. Dependence of impressions of three-dimensionality and impossiblity on relative size of local cues to their distance.

picture is likely to look either as a result of some error in drafting or as a set of flat geometrical figures, that is as a non-object similar to that in Fig. 1(b). In Fig. 2(b), on the contrary, the separation of contradicting elements seems nearly optimal: we see it strongly as some clearly three-dimensional structure, but an apparent contradictory arrangement of its elements produces on us a similarly strong impression of a visual paradox. Yet when the corners of the frame become still smaller (Fig. 2(c)), the three-dimensional as well as the impossibility impressions become again weaker. The figure looks like a flat frame with some not so important thickness, thus inconsistency of its set of vertices usually passes unnoticed.

It is important to add that the effect cannot be explained by simple change of dimensions (which would bring up explanations involving peripheral vision, necessity of eye-movements to grasp the whole, etc). In fact the dimensions of all three figures (Fig. 2(a), 2(b), 2(c)) are exactly the same.

The above effect and other similar ones call for more thorough investigations, involving appropriate psychological experiments with wider audience, similar, e.g., to that reported by Hochberg and Brooks [21] on connection between drawing complexity and impression of three-dimensionality (see also Young and Deręgowski [10]). Such experiments can reveal the relative importance of different 'spatial cues' for human interpretation processes, enriching our knowledge about the structure of our interpretation algorithms.

The second important word in the definition is, let us recall, '*interpretation*'. In the process of gathering the impression of three-dimensionality of the figure, we appropriately interpret different local and global depth cues to arrive at some over-all model of a three-dimensional structure of the object depicted. This process of interpretation is mostly unconscious and depends substantially on a huge set of memorized 'most likely' models of various familiar shapes. For, as was strongly pointed out by Gregory [6], every drawing can have infinitely many different interpretations as a three-dimensional object (i.e. it represents infinitely many spatially different objects). From this infinity, our perception process selects usually only one 'natural' interpretation. It is just this interpretation that is judged next as to its impossibility, not the original drawing itself. *The property 'to be an impossible figure' is not the property of the drawing alone, but the property of its spatial interpretation by some human being.* As was told above, any given drawing has infinitely many three-dimensional interpretations, some of them probably possible, some not. Such the drawing will appear in works of researchers on impossible figures not when any of its interpretations is impossible, but only when the one selected as 'natural' by some (or rather: by most) of *human perceivers* is impossible. It is an observation of fundamental importance to any reasonable analysis of impossible figures in particular, and spatial interpretation models in general. It indicates that phenomenon of impossible figures is unlikely to tell us anything new about geometry, but it can tell us at least something about human processes of picture interpretation. As such, this phenomenon is also important for the field of artificial intelligence, particularly, although not exclusively, for the robot vision research. It allows us not only to observe the fantastic although not yet understandable skill with which humans see three-dimensional arrangements of objects in a flat picture, but more importantly – to observe errors of this interpretation mechanism we want to model in our robots. It is often easier to learn the functional structure of an unknown mechanism from its errors than from its undisturbed functioning. It is even more promising than use of nonsense sentences in linguistics (the comparison devised by Huffman [2]), because impossible figures, having also possible interpretations, thus being not completely nonsense (see Section 5), still baffle our perceptual mechanisms, which do not notice these possible interpretations, even when we know them intellectually (note discussion of impossible triangle realization in Gregory [6]: *we can under-*

stand the possible structure of the figure, being still *unable to see* it, cf. also Section 3).

To illustrate the above, let us analyze the long-known Thiéry's figure [7]. Fig. 3 shows three of its different interpretations. It is most often considered as an ambiguous figure, i.e. such that it has two equally probably chosen interpretations, in this case they are: '*a cube seen from below-left with two flat appendages hanging down*', and '*a cube seen from top-right with two flat appendages standing up*'. This interpretation is also called '*convex/concave ambiguity*' (see e.g. Attneave [30], Ernst [26]) and as such it is closely related to Necker's cube, Schröder stairs and Mach illusion [6, 30]. It is the most strongly coming up interpretation of this figure. The next in sequence is the impossible interpretation: '*two cubes seen from different directions and improperly joined*'. This interpretation is often appearing as a 'first sight' impression, usually suppressed quickly in favour of the ambiguous one. But this figure has also the simple possible interpretation: '*a plate*

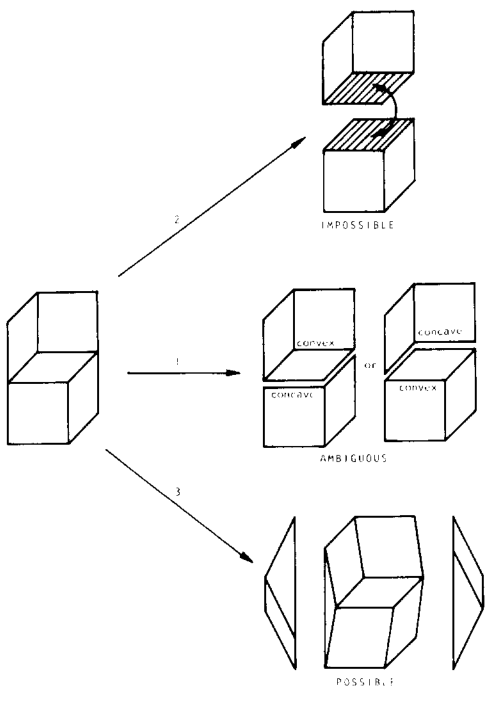

Fig. 3. Three types of interpretation of Thiéry's figure.

slantly cut in lower and upper parts', in this interpretation the central parallelogram is parallel to the surface on which the object lies supported by its backward hexagonal flat face. Fig. 3 shows also two side views of the object to facilitate the comprehension of its shape. In spite of its simplicity (compare it with the construction of the impossible triangle in Gregory [6]), practically no one of the spectators even considers such an interpretation of Thiéry's figure.

Huffman [2] tried to account for the variety of different types of interpretations of a given figure by introducing different geometrical classes of objects. He noticed that the impossibility of a figure depends on the assumed class of objects an interpretation of the figure is allowed to belong to. The given figure may be impossible in some such class but possible in another. Nevertheless, this formally (geometrically) defined classification of different interpretations does not resemble well the natural (to human perceivers) partition of the set of all interpretations of a figure into different types. For example, although the different types of interpretations of Thiéry's figure do belong to different Huffman classes, the set-theoretical and complexity hierarchy of these classes does not resemble the 'likeliness' of respective interpretations. The 'second likely' (impossible) interpretation belongs to the simplest Huffman class of '*trihedral solids*' (i.e. solids bounded by plane surfaces such that in every vertex only three different faces meet). The hardly visible possible interpretation belongs to the next (in complexity hierarchy) class of '*solids bounded by plane surfaces, not necessarily trihedral*', and the most probable ambiguous interpretation calls for a still wider class of '*plane-faced solids mixed with plane surfaces*'.

Moreover, examples exist that are in full conflict with this classification. Fig. 4(a) shows a figure whose two interpretations of different types belong to the same class of *trihedral solids*, and Fig. 4(b) shows an almost trivial example of a figure with two (at least) equally possible interpretations belonging to two different classes: *trihedral*

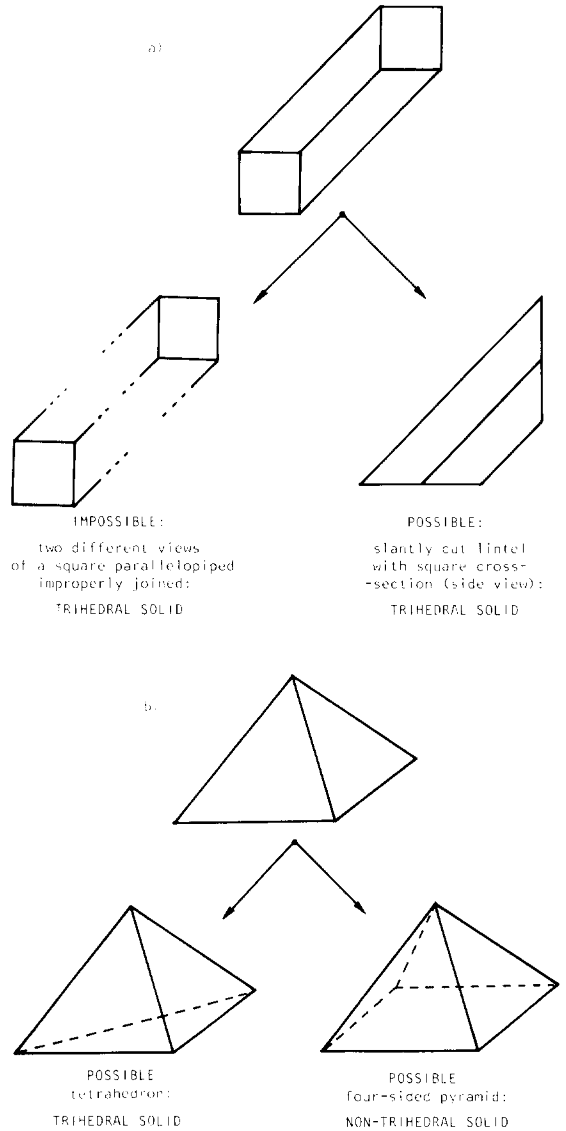

Fig. 4. Different types of interpretation can belong to the same geometrical class (a) or interpretation of the same type can belong to different classes (b).

solids and *non-trihedral plane-faced solids.* Egyptians and egyptologists would surely choose the second interpretation, whereas students of classical geometry – rather the first one.

Huffman [2] has made also some other assumptions about the nature of the objects depicted in drawings, e.g., a concept of the 'general position' of the object with relation to the observer. These assumptions were all aimed at restricting considerably the class of allowed interpretations, in order to make possible finding of simple impossibility criteria. Further discussion of these matters will be continued in Section 5.

3. 'Likely' and 'unlikely' figures

On closer examination, we can find also other kinds of figures, closely related to the impossible ones, although not fitting exactly into the definition from the previous section.

This was noted already by Huffman [2]. One kind of such figures he called '*unlikely*' – we will discuss them later. The second kind was not named by him, so we will call them here '*likely*' figures (in contrast to *unlikely* ones):

Definition 2. A *likely figure* is the figure whose interpretation, selected by an observer, is in fact impossible, but it is not noticed by the observer to be impossible.

Fig. 5 shows some examples of such figures. The first one (the 'impossible pyramid') was referred to several times by others, e.g. [2, 20, 31]. Fig 6(a) explains the usual interpretation of it, as a quite proper truncated pyramid with the triangular base. Yet such a pyramid is impossible – to be a projection of a real object, the edges *A*, *B* and *C*, being the lines of mutual intersection of the three plane faces of the pyramid, should intersect at a single point (when extended). As is evident from Fig. 6(a), they do not, thus such a pyramid is impossible. The discrepancy between points of pairwise intersection (S_{AB}, S_{BC}, S_{AC}) is fairly large, with the same order of magnitude as other clearly visible details of the pyramid, thus the 'likeliness' of the pyramid cannot be explained as being due to unnoticeability of minute deformations. It should be evident! But it is not, at least without actually drawing the lines extending the edges.

Similar reasons judge the figure in Fig. 5(b) as impossible: the front and upper planar surfaces

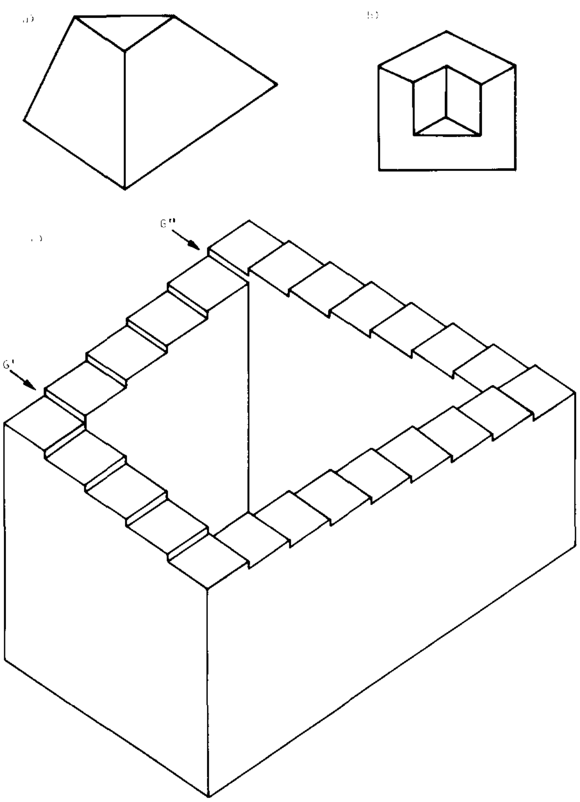

Fig. 5. Some examples of *likely figures*: impossible truncated pyramid (a), Huffman's corner (b) and large Penroses' staircase (c).

should intersect along a single straight line, yet they intersect in the figure along the two segments that do not lie on a single straight line. Again this impossibility is hardly noticeable, in spite of the fact that the segments involved are almost perpendicular.

The well-known Penroses' staircase (Fig. 5(c)), see [1, 2, 20, 26], is usually classified as an ordinary impossible figure. Yet many uninitiated people do not notice its impossibility, at least without longer consideration [20]. It happens especially with larger versions of the staircase (like that in Fig. 5(c)), with many steps possibly equally distributed among four sides of the staircase. Such large staircases are often considered as being quite normal – therefore they can be classified as intermediate between *impossible* and *likely* figures. To facilitate testing this phenomenon by the reader, we give here a recipe for designing staircases with any required shape. Denoting the number of stairs on the four sides of the staircase by $n_1, n_2, n_3, n_4 > 1$ (Fig. 7(a)), the shape of the parallelogram stair by appropriate lengths of segments of the sides of the circumscribed rectangle $a, b, a', b' > 0$ (Fig. 7(b)), the height of the step by $d > 0$ (Fig. 7(c)), and after conducting some geometrical reasoning, we conclude with:

$$\alpha = \frac{n_4 - n_2}{n_3 - n_1},$$

$$\alpha' = \frac{n_1 + n_2 + n_3 + n_4 - 4}{n_3 - n_1}\delta - \alpha,$$

where $\alpha = a/b$, $\alpha' = a'/b'$ and $\delta = d/b'$. The formulas allow us to calculate allowable combinations of numbers of stairs from required shape and height of the steps, or vice versa, thus providing means for construction of infinitely many different impossible staircases. Any particular staircase constructed in this way will have all stairs of equal shape and size. As can be derived from the formulas, minimal values of the number of stairs are 2, 2, 3 and 3, respectively. For these numbers we have $\alpha = 1$ and $\alpha' = 6\delta - 1$. When we assume $\delta = \frac{1}{3}$, we will get $\alpha' = 1$ also. Fig. 7(d) shows this version of the minimal staircase.

Properly speaking, the *likely figures* fall under the definition of impossible figures as formulated in the previous section. Nevertheless, they should be considered as a separate, even more puzzling class of the impossible figures variety. Indeed, the mind makes double error on them: not only it chooses an impossible interpretation but, in addition, erroneously judges this interpretation as correct. This second error blocks in most cases any chance for eventual recovery from the first one in order to find some truly possible interpretation. To separate likely figures from strictly impossible ones, the *Definition 1* in Section 2 should be appropriately supplemented, e.g. with the sentence:

"..., and the impossibility of this interpretation is immediately seen by the observer".

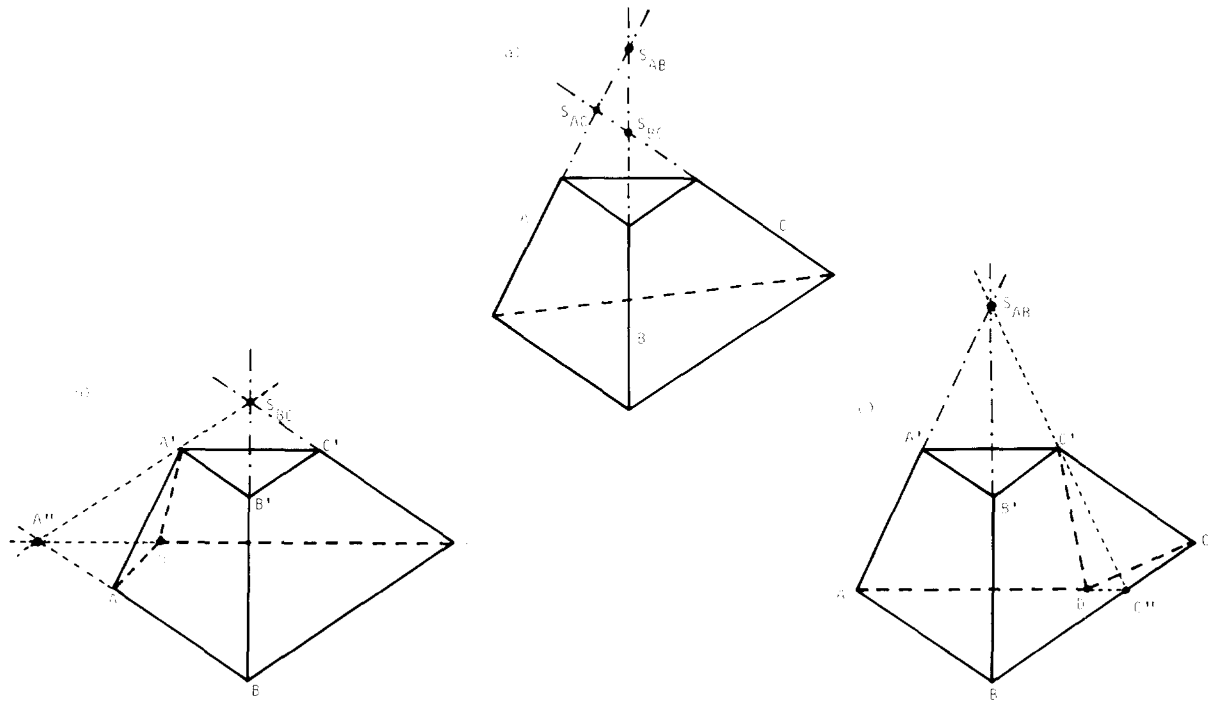

Fig. 6. Usually assumed interpretation of impossible pyramid and proof of its impossibility (a), and construction of two families of possible interpretations of the same pyramid (b, c).

The possible interpretations of the impossible pyramid are in fact numerous and quite simple. Figs. 6(b) and 6(c) show two families of them. In both cases it is required to split up the invisible back side of the pyramid into two sides. The point D can take any position on the line $A''C$ and AC'' (except that in the first case it should not appear on the left of the line AA', because then the view of the pyramid would change). In extreme cases, D can fall at the point C in Fig. 6(b) and at the points A or C'' in Fig. 6(c). Then the base of the pyramid remains triangular, and if $D = C$ or $D = A$, the back side consists of two triangles also. In the second case (Fig. 6(c)), when $D = C''$, the figure becomes the triangular truncated pyramid $ABC''A'B'C'$ with the flat appendage $CC'C''$ attached to the $BB'C'C''$ face. Just this appendage is now responsible for deviating the edge CC' from its way to the intersection point S_{AB}. The smaller the appendage, the more 'possible' is the pyramid, until in the limit $C = C''$, the appendage vanishes and the pyramid becomes at last well-behaved.

The Penroses' staircase has also a possible interpretation – it suffices to get over an illusion that the staircase is closed and allow it to have a gap at the point G' or G'' (Fig. 5(c)). It was noted by the Penroses themselves – they even made an appropriate plaster model and photographed it [1, 26], constructing probably the first 'impossible object', at least before Gregory [6].

Another sort of figures, closely related to the likely figures, can be called '*damaged figures*'. Fig. 8 shows two examples of them. They can be classified as *likely*, because geometrically they are impossible. The first one is impossible by the same argument as Fig. 5(b), the second as being some variety of the well-known 'three-stick clevis' figure type [22, 23, 2, 6, 20] (cf. Fig. 14(h)) – but they are rarely judged as impossible. Observers usually add mentally a missing edge to them, possibly with a comment that the drawing (not the object!) is 'unfinished' or 'damaged' (hence the name). It reflects everyday-life situations where very often (e.g. because of lighting conditions) edges of

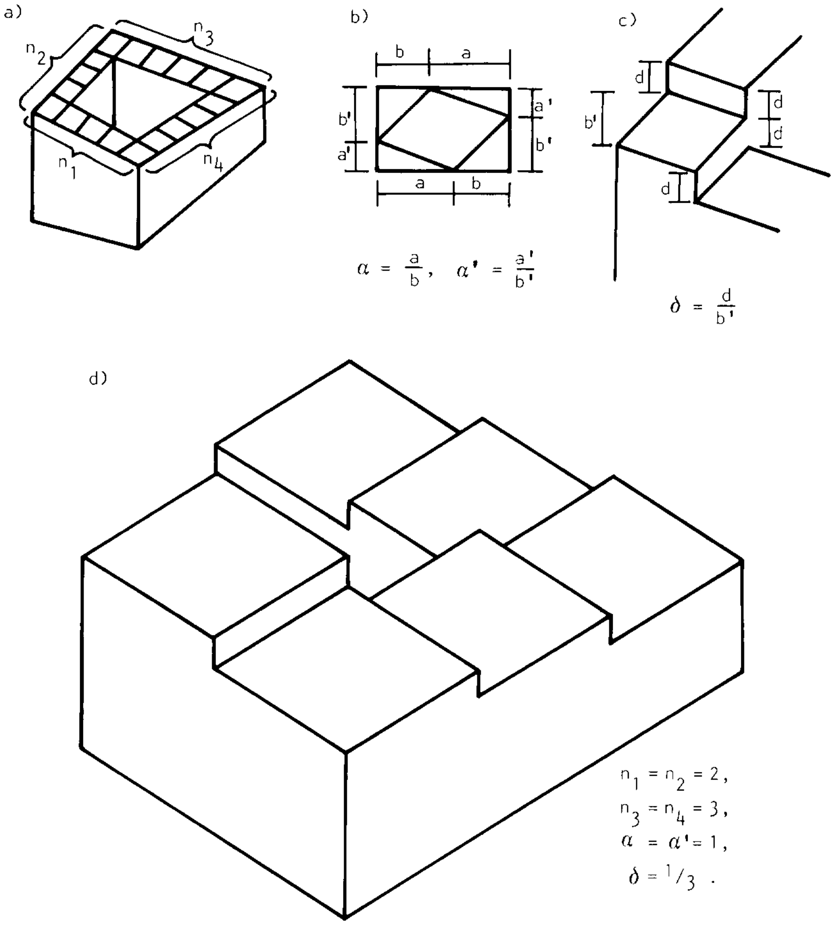

Fig. 7. Parameters of impossible staircases ((a), (b), (c) – see text), and the minimal staircase (d).

objects are missing and should be assumed to exist on the basis of knowledge rather than visual evidence (cf. Clowes [5]). The effect has caused many a headache to computer-vision researchers, its abundance being often unexpected to them; our brain corrects images unconsciously, so we rarely realize how often and how sometimes extensive these corrections are needed and actually occur. Eventually computer programs were also endowed with (as yet rudimentary) abilities to complete missing informations, usually by fitting of object models to processed image data (see e.g. Roberts [11] and Clowes [5]).

Still another cause of likeliness can be probably extracted from the example in Fig. 9. Everyone sees there simply a normal and perfect cube. Yet practically it is never possible to see any three-dimensional cube in this manner. Seeing it similarly would require to look from an infinite distance, with the centre of vision (projection) displaced far away to the right and up from the cube. Nevertheless, looking at this drawing we see nothing wrong, in spite of the fact that really we interpret it wrongly as a perfect cube seen from near distance and in the centre of the visual field. In doing this, we stick to some generally approved convention, widely used in technical drawing (a so-called axonometric perspective or parallel projection) as well as in some varieties of visual arts (Middle-Ages, esp. Eastern art, children and primitive art: Arnheim [27], Raushenbakh [28]). Also almost all drawings in this paper use this

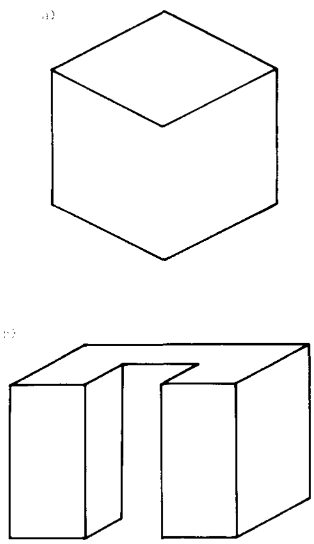

Fig. 8. Missing edges or impossible figures?

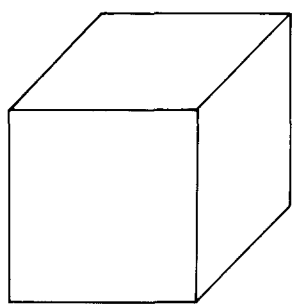

Fig. 9. A *conventionally likely* cube.

convention (although their impossibility features are not due to this fact). Universal use of this convention and universal unnoticeability of its inconformity with the reality of central projection (linking three-dimensional world with flat images on our retinas) indicates presence of some fundamental feature of our visual interpretation mechanism. It seems that it is an effect of the so-called size and shape constancy mechanisms of visual perception, see [27, 28]. Deeper analysis of this problem goes beyond the scope of this paper (but see Section 5).

Concerning '*unlikely*' figures, Huffman [2] had not defined them, showing only some examples. Taking some of his examples (Fig. 10(b), (e)), and some other (Fig. 10(a), (c), (d)) we might define unlikely figures (in wide sense) as drawings that seem impossible, although they have easily noticeable possible interpretations. Usually the perceiver sees firstly an impossible interpretation (cf. Fig. 4(a)), but aroused by its impossibility, quickly finds another, possible interpretation. So defined unlikely figures are simply the impossible figures with low '*degree of impossibility*'. The boundary between strictly impossible figures and unlikely figures defined as above is fuzzy, depending heavily on the experience and spatial imagination of the perceiver. In the limit (see the next Section) one might conclude that eventually all impossible figures are only unlikely (as all of them have ultimately some possible interpretations). Therefore, such unlikeliness definition is not very useful as a discriminating tool, and should be replaced by some measure of degree of impossibility.

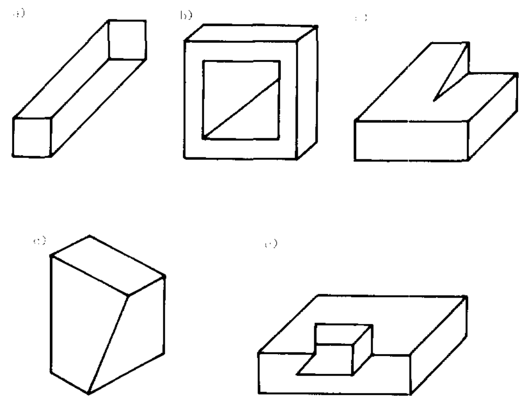

Fig. 10. *Unlikely figures* (a–e) and *unlikely figures in strict sense* (d, e).

Nevertheless, Figs. 10(d) and 10(e) show examples of figures constituting a class more markedly distinguishable from impossible figures. This class has not yet been considered or investigated, and comparatively few examples have been found. The definition of such figures (we propose to call them '*unlikely figures* in the strict sense') can be formulated as follows:

Definition 3. An *unlikely figure* (in the strict sense) is the figure whose interpretation, selected by an observer, is in fact possible, but is considered by the observer as impossible.

Here the interpretation mechanisms make an error at the phase of verification of the interpretation. The interpretations usually given for Figs. 10(d) and 10(e) by human observers can be formulated as '*a skewed die*' and '*a plank with slantly cut cavity at the edge*'. In spite of their possibility, they are usually treated as improbable or even impossible to make without violation of some essential features of these interpretations (e.g., planarity of the faces of the die). Even convincing arguments in favour of their possibility cannot destroy completely a vague feeling of their unlikeliness. The inclusion of the figures from Fig. 10(d), (e) into the class of unlikely figures in wide sense was justified by the fact that their interpretations, being at first claimed to be impossible, can be after some considerations approved at last as possible.

The above-defined notions can be arranged neatly into the diagram of Fig. 11 – it shows completeness of our classification with respect to impossibility features of the interpretation of the figure. The *damaged figures* are of course included in the *likely* class.

The picked up interpretation: is / is judged as	possible	impossible
possible	possible figs.	likely figs.
impossible	unlikely figs. (strict sense)	impossible figs. (strictly)

Fig. 11. Impossibility classes of figures.

4. Are impossible figures possible?

If the answer to the above question was 'no' (that is, 'impossible figures are not possible'), it would mean that there are no impossible figures, i.e. every figure (also that considered impossible) has in fact some possible interpretation(s). Curiously enough, the answer 'yes' can be also interpreted to the same effect: 'impossible figures are possible', that is all impossible figures are possible to construct, thus every impossible figure has in fact some possible interpretation(s). As a result, no matter what the answer is, the conclusion becomes the same[2]. Is this conclusion really true?

As has been shown, for many 'very impossible' figures there were indeed found possible interpretations (i.e. impossible objects were constructed):

- the Penroses' staircase (see [1, 26] and Fig. 5(c),
- the impossible triangle (Gregory [6] invented an open 'fork' object, Fig. 14(c); Koleichuk [25] invented the whole family of closed, but curvilinear-edged triangles),
- the impossible pyramid (Fig. 6),
- the Thiéry's figure (Fig. 3), etc.,

see also Figs. 4 and 10. Therefore, all these impossible figures should be considered to be at most unlikely figures (in wide sense), i.e. figures looking impossible although having possible interpretations. In the course of becoming more and more acquainted with diverse impossible and unlikely figures one easily finds that for more and more figures considered to be impossible one can devise possible interpretations, although may be a little complicated and uncommonly shaped.

But is it true in general? Let us have a look at Fig. 12. It is a trivial geometrical fact: if we consider any line AB on a flat picture of a figure, then any line having ends on the lines SA and SB and lying entirely in the plane SAB (where S is the centre of the projection) will have the same projection AB on the picture plane. Repeating the same procedure for all lines of the figure, we can obtain a three-dimensional interpretation (in a form of some 'wire model') of any figure, including all impossible ones as well. The model can be disconnected or connected, depending on whether

[2] Of course, it is a typical Humpty-Dumpty reasoning: the reader can esaily notice the use of the word 'possible' in two slightly different meanings.

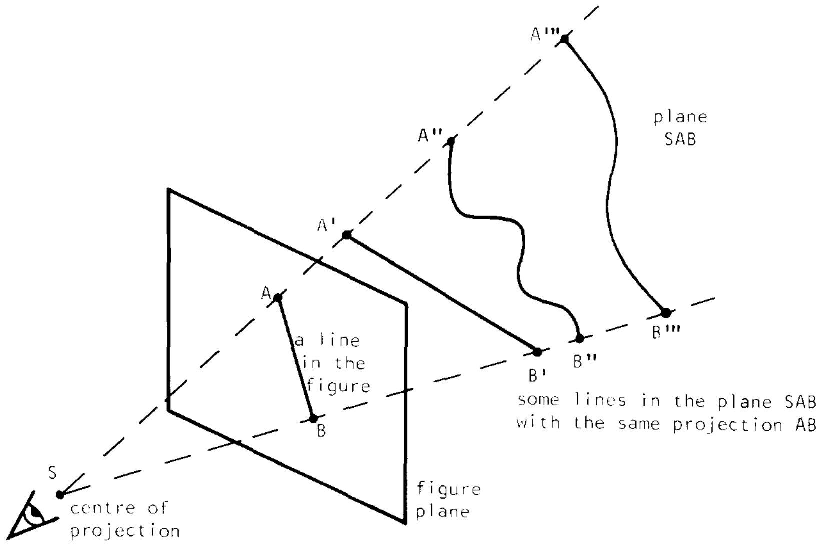

Fig. 12. Any line lying on the plane SAB and having ends at the straight lines SA and SB has the same projection AB.

we have taken care to place the ends of the wires modelling connected segments in the figure at the same point along appropriate SA (or SB) line. With a little effort, we can spread surfaces between nearby wires and make some wires to become edges of solids. There are infinitely many possibilities!

With this excellent method of construction of impossible objects, the problem of impossibility might seem to vanish. But it is not quite so – as we have seen in Section 2, the impossibility property is not the property of the drawing, but the property of its interpretation. The above construction method serves as another proof of this observation: it states that in fact the impossibility property even *cannot* be the property of the drawing alone. Nevertheless, the interpretations of impossible figures still remain impossible, and the question why observers select just them, instead of some offered by the method above, still remains unanswered. This indicates once more that the problem belongs to the field of psychology of vision rather than to geometry.

For what purpose can geometry, or any other mathematical formalism be used in analysis of impossible figures? It seems that it can serve here two purposes:

(1) To enumerate systematically figures of certain structures or classes (not necessarily including only impossible figures, see Cowan [8, 9]) for the purpose of putting some order in them, facilitating reference and selection of appropriate examples for experiments (e.g. psychological).

(2) To serve as a formal language of formulating models of interpretation mechanisms and describing interpretation classes of figures (for this purpose it was mostly used by Huffman [2]).

Therefore, formal models serve merely as an auxiliary tool to describe essentially psychological phenomena. One cannot expect that purely mathematical analysis of drawings, however sophisticated but without any reference to a human perceiver, can result in discovering principles of human three-dimensional interpretation, including an explanation of the phenomenon of impossible figures.

The above considerations can be well summarized by the words of the Carroll's White Queen. After some practice, one should not have any troubles with *believing impossible things*, in both senses of the words: believing they exist, i.e. are all in fact realizable, and believing their *impossibility*, i.e. that the problem of their impossible interpretations also exists and deserves interest.

5. How do we interpret flat projections?

In this section we try to formulate some observations on the structure of human visual interpretation processes, deducible from the discussion of the impossible figures and related phenomena. It does not mean, of course, that the conclusions drawn below are universally valid, or uniquely following from the facts (can there be any really firm facts in the world of illusions?), or new (some of them surely were or can be formulated on the basis of other observations as well). We formulate some hypotheses and conjectures which can be used as points of departure for further research and psychological experiments, or can confirm similar results found in other ways (see e.g. Kanade [18] and Draper [19]).

5.1. Two-stage interpretation

It seems that the interpretation process can be divided into two main stages (or co-operating subprocesses):

(1) Analytical interpretation based mainly on local depth cues of all various sorts.

(2) Global synthesis, including verification, adjustment and correction of local evidence, and based at least in part on fitting to models (memorized general patterns).

An occurrence of impossible figures and other related types of figures (see Section 3 and Fig. 11) can be systematically explained in terms of errors of these stages, namely:

- *impossible figures*: indicate an error of the first stage (suggesting an impossible instead of possible interpretation), detected (found to be contradictory) by the second stage, but not corrected there;

- *impossible figures with low degree of impossibility* (unlikely figures in wide sense): indicate an error of the first stage (suggesting an impossible interpretation as a first-step hypothesis), but then suitably corrected by finding another (possible) interpretation, equally consistent with the data (usually after repeating the first stage with a different 'tuning');

- *unlikely figures* (in strict sense): indicate an error of the second stage (judging the possible interpretation as impossible), probably due to lack of appropriate and simple enough object model fitting the interpretation;

- *likely figures*: indicate an error of the first stage (producing an essentially impossible interpretation), followed by an error of the second stage (approval of this impossible interpretation as valid); for *damaged figures* (Fig. 8) this approval is caused by previous correction or completion of the input data (according to the hidden assumption that the input data are allowed to be incorrect or incomplete);

- *possible figures*: indicate a proper work of both stages, producing some unique and spatially realizable interpretation, looking 'natural' for the observer.

The work of the first stage is based, seemingly, on more or less local detection of different depth cues and elementary local models (characteristic fragments of spatial objects, like various corners, ends of lintels, etc.) [6, 8, 9, 12, 13, 15, 21, 27]. The role of oblique lines in suggesting depth and inclination of surfaces should be also stressed here (Fig. 13(b)). It is usually not taken into account by robot vision researchers [4, 11–15, 17–19] although it is considered as a basic device by artists [27, 28].

The work of the second stage includes probably the two basic schemes: fitting into agreement these local hypotheses on more and more global context (e.g. by means of a so-called relaxation labeling process guided by various heuristics, see [29] on the use of this process in computer vision) and matching generalized object models with the data (see [11] for a classic example of computer implementation of this process). During the execution of these processes, initial hypotheses may be made more precise, or may be changed (either 'by force' or after repeating the local analysis with a different 'tuning'), or supposedly lacking data can be added to the input. Sometimes this correcting of reality, although indispensable in real-life situations, leads to an error, indicated by some examples of likely figures, especially of the *damaged* variety. Another example is provided by the impossible pyramid (Fig. 5(a) and 6(a)), where non-intersecting lines are mentally 'straightened', the more easily because their eventual intersection is only presumable, being not visually present in the drawing.

The detailed structure of these stages and their mutual interaction is rather complex and its deciphering is still a great challenge to artificial intelligence and psychological research. The analysis of the phenomenon of impossible figures seems to be profitable in significant amount here.

5.2. Interpretation assumptions

One of the most puzzling aspects of our spatial interpretation mechanisms is their ability to filter out so easily the infinite bulk of allowable interpretations. They bring forth usually only one interpretation with convincing feeling of 'naturalness'. It has been also discussed by Kanade [17, 18], who wondered how to express formally the notion of 'naturalness', necessary to filter out so usually great numbers of geometrically consistent interpretations generated by his geometrical algorithms. The problem cannot be solved only by means of straightforward fitting in stored models with the interpretations (see previous subsection). Any practical competence in the sufficiently rich visual world would require such enormous numbers of different models that their straightforward storage and exhaustive searching (like that used by Roberts [11]) soon becomes intractable.

It seems that the process of search through the net of possible interpretations is guided by some rather general *'interpretation assumptions'*, variously formulated by psychologists for long ago [6, 21, 27]. For our purposes they can be formulated as follows:

(1) *Simplicity* assumption: the interpretation should be as simple as possible, preferably not more complex than the drawing.

(2) *Minimal change* assumption: geometrical features and relations present in the drawing (e.g. straightness, parallelism, intersection relations, etc.) are preserved also in the three-dimensional interpretation (provided it does not violate the simplicity assumption too much).

(3) *General position* assumption: the object is depicted in the drawing from a nonsingular point of view, i.e. such that a slight change of the assumed point of view (centre of projection) does not change significantly either the structure (e.g. topological) of the drawing or important features of its elements (e.g. straightness, parallelism, etc).

These assumptions are not fully independent, e.g. existence of a curved line in the object, situated such that it produces a straight projection in the drawing (see Fig. 12) violates all of them.

Discrimination of various geometrical classes of three-dimensional objects the spatial interpretation of the drawing is assumed to belong to (see discussion in Section 2) might be also presented as leading to formulation of some other interpretation assumptions. Although it is partially useful for this purpose, in general this approach, besides offering an excuse for over-simplification of many formal approaches to the problem (most algorithms proposed so far work well, if not only, for the simple objects class of trihedral solids [2–5, 11–15]), seems to offer very little (cf. Fig. 4).

On the contrary, the assumptions listed above constitute truly important features of classes of interpretations our vision mechanisms are trying to fit in with the analyzed drawings. It seems that finding the answers to the question *why* and *when* these assumptions are made by human observers is crucial for achieving any success in full explanation of the impossibility phenomena.

Classic examples illustrating the role of the simplicity assumption are recalled here in Fig. 13, see also [21, 27]. Elements and relations simple in the drawing remain equally simple (or even become simpler, e.g. acute and obtuse angles are interpreted as (projections of) right angles, see Fig. 13) in the spatial interpretation. Interpretations more complex than the drawing – with invisible gaps

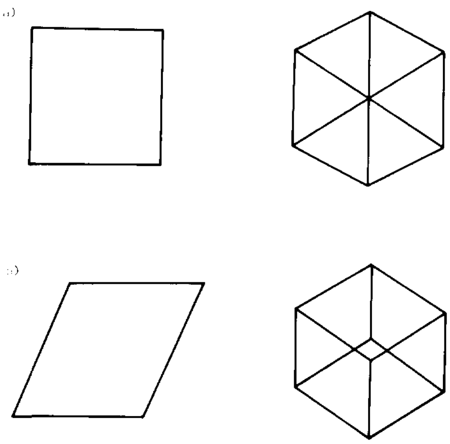

Fig. 13. A role of simplicity in selection of interpretations: figures under (a) are simpler as flat, whereas that under (b) are simpler as spatial (note also the role of oblique lines in the parallelogram).

(Penroses' staircase, Fig. 5; Gregory's realization of the impossible triangle [6]), with curved lines looking straight in the drawing (curvilinear realization of the impossible triangle by Koleichuk [25]), with additional hidden lines (e.g. impossible pyramid, Fig. 6), etc. – are ruled out as improbable, or are not taken into consideration at all, just at the first stage.

Also unlikely figures in strict sense seem to be explicable as a result of discarding interpretations more complex than the drawing. In them, the possible interpretation, although it has been found, seems to be so complex, and just unlikely and unfamiliar to the observer, that its evaluation as impossible or at least very doubtful looks well justified.

To explain origins of these assumptions, let us observe that any projection of three-dimensional objects on the two-dimensional picture plane inevitably deforms shapes of the original objects – parallel lines become convergent, right angles become acute or obtuse, edges of equal length become unequal, etc. In short – the projection becomes more complex than the original. It is therefore quite reasonable to look for spatial interpretation of a drawing among objects simpler than this drawing.

Moreover, the ability of three-dimensional interpretation of flat pictures (monocular depth perception) is a relatively novel, specifically human ability. In natural circumstances we use far more reliable means of depth perception: binocular parallax and motion parallax. Just this is the place where the third (and the second) interpretation assumption is rooted: we do not expect interpretations which would produce greatly different appearance (from that seen in the picture) when viewed from slightly different eye position. Perceiving any real object, we always see it from several slightly different positions – either simultaneously (binocular parallax) or in close time instants (motion parallax – small head movements suffice). Therefore it is practically impossible to see any object only from just this special position, from which its elements are so aligned as to suggest quite another interpretation (of the object and its elements) than that evident from slightly shifted viewing direction. Such uncommon and unstable viewing points were thus not accounted for in interpretation procedures of our brains. The same procedures were then used to perform the task of interpreting flat pictures as well. In consequence, we (or rather these subconscious interpretation procedures) assume that the drawn figure is also seen from the more common, or 'general' position, not from this special one hiding meticulously the real structure of the object. Having in disposal only this one view, we assume that all relevant information is present, and we usually do not try to hypothesize any additional data, not seen in the picture.

On the contrary, we take various regularities present in the drawing, e.g. parallelism, alignment and contact relations between lines, equality of lengths and angles, etc., as not accidental, but indicative of corresponding regularities in the spatial interpretation (the second interpretation assumption, see also Kanade [18]).

Therefore we see the Penroses' staircase closed (Fig. 5(c)), not taking into account the possibility of occurrence of a gap in it; we do not attempt to imagine the invisible non-intersection of edges in the impossible pyramid (Fig. 6(a)) or the carefully aligned cutting in the vertical stripe of the impossible cube (Fig. 14(d)). However, exceptions can occur to these rules, especially when:

– the missing data are simple and are of the type commonly being lost in real-life circumstances (*damaged figures*, Fig. 8), or are nothing but a small difference between the figure and some well-known typical object (some *damaged figures*: Fig. 8(a); hidden lines: Figs. 4(b), 6(a); *conventionally likely* figures: Fig. 9, etc.);

– the detection of impossibility of the interpretation forces us to widen our interpetation range and try less probable (less natural) possibilities (*unlikely figures in wide sense*: Figs. 10(a), (b), (c)).

The regularity-conservation rule and the habit to look for only typical views, discarding special alignment circumstances, seem also responsible for the *unlikely figures* phenomenon (Fig. 10). There just that carefull alignment of certain features and regularities, usually seen (or suggested by the drawing) to be unrelated or incompatible, produces the unlikeliness feeling, resulting in the classification of the corresponding interpretation as impossible.

Nevertheless, the general position assumption, especially taken in isolation, although important, cannot explain all phenomena of impossible figures. E.g., it does not explain fully the impossible pyramid (Fig. 6(a)). This figure has the same appearance, with the same basic structure, when viewed from considerably wide range of viewing directions. Also many other impossible figures have possible realizations without accidental alignments, even such figures as that in Fig. 14(a).

5.3. Impossibility sources and impossibility detection

Impossible figures allow us to isolate the kinds of local features (partial interpretations) that are tested for conformity in the course of verification of spatial interpretations. Arranging these local features in contradictory manners produces the impossibility. Therefore impossible figures constitute a rich source of information about local interpretations and their mutual relations that play the significant role in organization of human picture interpretation processes. The impossible figures can be said metaphorically to be just their most expressive portraits. The three seemingly basic '*sources of impossibility*' are enumerated in Fig. 14.

The strongest impossibility seems to be produced by violating the *figure/background distinction* (Fig. 14(a)). For impossible figures exhibiting this sort of impossibility it is hard to find any good possible interpretation. Feasible interpretations appear rather degenerated, in the sense that they allow realization in the form of wire models rather than solids suggested by the drawing. A new, so far unnoticed '*self-shadow/projected shadow*' contradiction shown in Fig. 14(b) can be classified as a variant of the figure/background one (in this example combined also with the '*plane twisting*' contradiction discussed below, Fig. 14(e)).

Also strong, although markedly weaker contradiction is produced by different *estimation of depth relations* between figure elements (Fig. 14(c), (d)). It seems to be the most popular device in devising impossible figures. In the impossible triangle (Fig. 14(c)), corner configurations (1) and (2) suggest that the beam end (1′) is farther than the beam (1-2), while the beam end (2′) is nearer than (1-2). In the full triangle both ends are joined, becoming, then of the same depth (1′ = 2′) which contradicts those suggestions. The Gregory's construction [6] of the impossible object corresponding to the impossible triangle follows strictly this interpretation – he produced the object with three beams arranged as at the left side of the figure and then photographed it from such a position that the ends (1′) and (2′) became visually joined (in order to produce exactly the right-side appearance, an additional cutting in the end (2′) should be made, see Gregory [6]). Fig. 14(d) shows a similar situation: the overall construction suggests that the strip (1) is nearer than the strip (2) (left side), whereas the local configuration encircled in the right-side drawing indicates the opposite. It is also

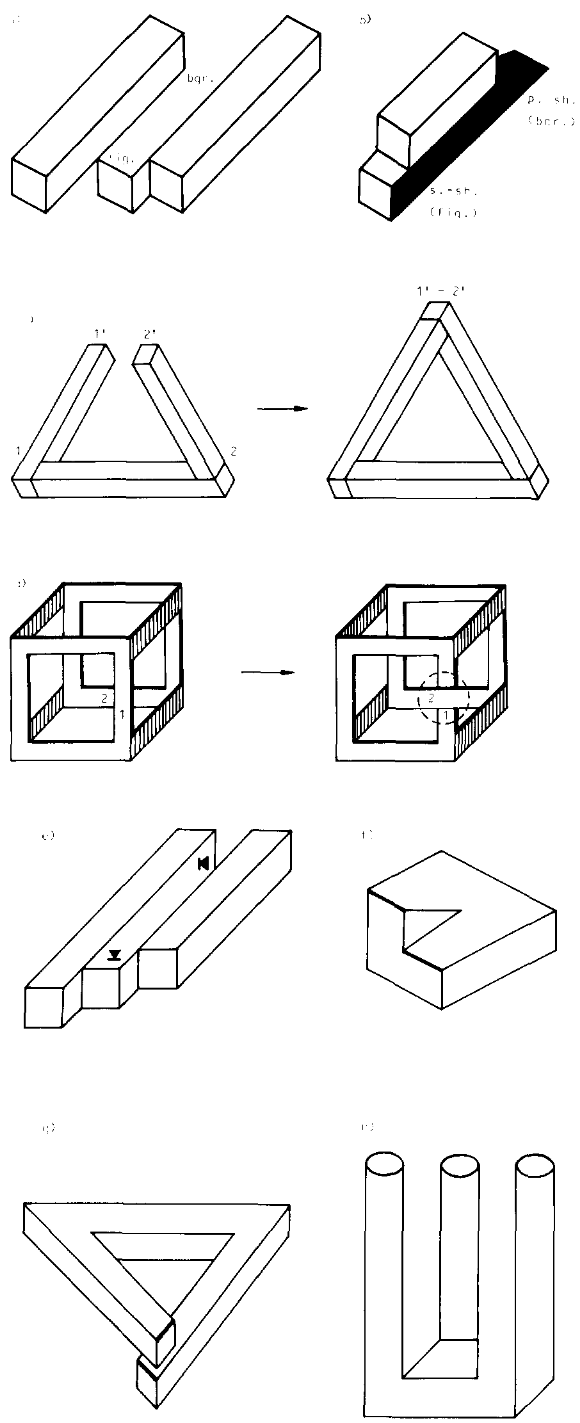

Fig. 14. Main types of contradictions between partial interpretations in impossible figures: figure/background (a); projected shadow (background)/self-shadow (figure) (b); different depth estimation for the same element (c, d); horizontal plane/vertical plane (e); warped plane (f, g); mixed case (g).

comparatively simple to construct a corresponding impossible object (Hyzer [24]). The motif inspired also the late Dutch artist M.C. Escher [26].

Another impossibility source, weaker and easier to resolve than the previous two, can be called '*plane twisting*' (Fig. 14(e), (f), (g)). In Fig. 14(e), the nearer end of the middle strip is suggested to be horizontal (being an upper surface of a horizontally lying beam), whereas the farther end is suggested to be vertical (being a side surface of another similar beam). It is possible only if it has been twisted, but all the context in the drawing suggests strongly that all strips are strictly planar. Figs. 14(f) and 14(g) show a similar effect: the upper surface is suggested to be planar (by its straight-line edges and the context, remind the second interpretation assumption from Section 5.2), whereas a level difference between near parts of it (thicker edges) indicates that it cannot be a single planar surface. The figures become eventually possible if we allowed this surface to be warped.

In many impossible figures, different impossibility sources occur, mixed in various ways. E.g., in the well-known three-stick clevis or 'blivet' figure [22, 23, 6, 10, 20], Fig. 14(h), in three places a figure/background contradiction occurs, similarly in three places a flat strip twists into a cylindrical surface, and also some interpreters notice contradictory depth estimation for the position of the middle prong (Gregory [6]).

It should be stated here that the geometrical interpretation of the impossibility of Fig. 14(f) (and other similar ones), stating that it is impossible because two planar surfaces cannot intersect along two different lines [2] (marked as the thicker edges in the drawing), although geometrically valid, is not valid visually. Our visual system seemingly does not make geometrical deductions of this sort. It can be additionally supported by the case of the impossible pyramid (Fig. 5(a)). Here, to detect its impossibility, also some geometrical reasoning is necessary; moreover, it requires drawing some auxiliary lines (Fig. 6(a)). The fact that our visual interpretation system does not make this sort of deductions seems responsbile for our astonishing failure in detection of impossibility of that pyramid. See also the figures depicted in Figs. 5(b) and 8(a), where the same geometrical contradiction as above usually does not produce the feeling of impossibility – it is not *seen* as a contradiction.

Certain impossible figures can be sometimes detected to be impossible on the basis of contradictory edge interpretations. E.g., in Fig. 14(a) the third long edge from the right is at the far end an obscuring edge, whereas at the near end – a crack edge (Waltz [13]). Similarly, the second long edge from the left in Fig. 14(e) is simultaneously a crack edge and a convex edge. These edge features were favoured by computer-vision researchers and most of scene interpretation algorithms and computer programs have been based on them [2–5, 12–15, 17–19]. However, on the one hand this edge interpretation method is unable to detect great many impossible figures (Figs. 2, 5, 14(c), (d)), and on the other hand it seems that humans base their visual analysis rather on hypotheses about *surfaces* positions and orientations (the second impossibility source) and their features (the third impossibility source). A kind of this surface-oriented approach to the interpretation of drawings, called 'sidedness reasoning' has been proposed recently by Draper [19] on the basis of his analysis of failures of those edge-oriented approaches.

Conclusions

The case of impossible figures can teach us many things about principles and mechanisms of human interpretation (especially spatial) of pictures, helping us in the endeavour of endowing computers with similar abilities. It also teaches us some more general methodological lesson about limitations of formal approaches to modelling of human abilities.

Looking from more philosophical point of view, we can see once more that intelligent beings, like humans or future artificially-intelligent computers, comprehend the surrounding world not directly, but always according to their internal 'world model' – a complex net of knowledge, beliefs and habits. The source of impossible figures phenomenon lies in limiting effects of these habits. We are so used to our 'normal' circumstances – this allows us to function within them easily – that when confronted with something quite different, we still try to measure the new thing with the old rule. In favourable circumstances, we strike upon contradictions that make our habitual interpretation impossible. In that case we have a chance to get out of our narrow-mindedness, and widen our comprehension of reality a little. Otherwise, when things have been hidden more subtly, we do not notice other possibilities – all remains so 'likely' as usual . . .

The impossible gives us a chance – the 'likely' leads all too often into a blind alley.

Acknowledgments

The research reported here was supported by the Research Programmes Nos. 10.4 and 06.9.

References

[1] L.S. Penrose and R. Penrose, "Impossible objects: a special type of visual illusion", *Brit. J. Psychol.*, Vol. 49, No. 1, 1958, pp. 31–33.

[2] D.A. Huffman, "Impossible objects as nonsense sentences", in: B. Meltzer and D. Michie, eds., *Machine Intelligence 6*, Edinburgh University Press, Edinburgh, 1971, pp. 295–323.

[3] D.A. Huffman, "Realizable configurations of lines in pictures of polyhedra", in: E.W. Elcock and D. Michie, eds., *Machine Intelligence 8*, Ellis Horwood Ltd., Chichester, 1977, pp. 493–509.

[4] M.B. Clowes, "On seeing things", *Artificial Intelligence*, Vol. 2, No. 3, 1971, pp. 79–116.

[5] M.B. Clowes, "Scene analysis and picture grammars", in: *Machine Perception of Patterns and Pictures*, The Institute of Physics, London and Bristol, 1972, pp. 243–256.

[6] R.L. Gregory, *The Intelligent Eye*, Weidenfeld & Nicolson, London 1970.

[7] A. Thiéry, "Über geometrisch-optische Täuschungen", *Philosophische Studien*, Vol. 11, No. 3, 1895, pp. 307–370.

[8] T.M. Cowan, "The theory of braids and the analysis of impossible figures", *J. Math. Psychol.*, Vol. 11, No. 3, 1974, pp. 190–212.
[9] T.M. Cowan, "Organizing properties of impossible figures", *Perception*, Vol. 6, No. 1, 1977, pp. 41–56.
[10] A.W. Young and J.B. Deręgowski, "Learning to see impossible", *Perception*, Vol. 10, 1981, pp. 91–105.
[11] L.G. Roberts, "Machine perception of three-dimensional solids", in: J.T. Tippet *et al.*, eds., *Optical and Electro-optical Information Processing*, MIT Press, Cambridge, MA, 1965, pp. 159–197.
[12] A. Guzman, "Decomposition of a visual scene into three-dimensional bodies", *AFIPS Proc. of FJCC*, Vol. 33, Thompson Book, New York, 1968, pp. 291–304.
[13] D. Waltz, "Understanding line drawings of scenes with shadows", in: P.H. Winston, ed., *The Psychology of Computer Vision*, McGraw-Hill, New York, 1975, pp. 19–91.
[14] A.K. Mackworth, "Interpreting pictures of polyhedral scenes", *Artificial Intelligence*, Vol. 4, 1973, pp. 121–137.
[15] A.K. Mackworth, "How to see a simple world: an exegesis of some computer programs for scene analysis", in: E.W. Elcock and D. Michie, eds., *Machine Intelligence 8*, Ellis Horwood Ltd., Chichester, 1977, pp. 510–537.
[16] Z. Kulpa, "Are impossible figures possible?", Institute of Biocybernetics and Biomedical Engineering Internal Reports, Warsaw, 1980 (unpublished).
[17] T. Kanade, "A theory of Origami world", *Artificial Intelligence*, Vol. 13, 1980, pp. 279–311.
[18] T. Kanade, "Recovery of the three-dimensional shape of an object from a single view", *Artificial Intelligence*, Vol. 17, 1981, pp. 409–460.
[19] S.W. Draper, "The use of gradient and dual space in line drawing interpretation", *Artificial Intelligence*, Vol. 17, 1981, pp. 461–508.
[20] Z. Kulpa, "Obiekty nieistniejące" ("Nonexisting objects", in Polish), *Problemy*, No. 9 (366), 1976, pp. 20–22.
[21] J. Hochberg and V. Brooks, "The psychophysics of form: reversible-perspective drawings of spatial objects", *Amer. J. Psychol.*, Vol. 73, 1960, pp. 227–354.
[22] D.H. Schuster, "A new ambiguous figure: a three-stick clevis", *Amer. J. Psychol.*, Vol. 77, 1964, p. 673.
[23] J.O. Robinson and J.A. Wilson, "The impossible colonnade and other variations of a well-known figure", *Brit. J. Psychol.*, Vol. 64, No. 3, 1979, pp. 363–365.
[24] W.G. Hyzer, "Plenty of chances to go astray in photographic interpretation", *Photo Methods for Industry*, Jan. 1970, pp. 20–24.
[25] V.F. Koleichuk, "Nevozmozhnye figury?" ("Impossible figures?", in Russian) *Tekhnicheskaya Estetika*, No. 9, 1974, pp. 14–15.
[26] B. Ernst, *The Magic Mirror of M.C. Escher*, Ballantine Books, New York, 1976.
[27] R. Arnheim, *Art and Visual Perception – Psychology of the Creative Eye*, University of California Press, San Diego, 1974 (Polish translation: Artistic and Film Publ., Warsaw, 1978).
[28] B.V. Raushenbakh, *Prostranstvennye Postroyenya v Zhivopisi* (*Spatial Constructions in Visual Art*, in Russian), Nauka, Moscow, 1980.
[29] L.S. Davis and A. Rosenfeld, "Cooperating processes for low-level vision: a survey", *Artificial Intelligence*, Vol. 17, 1981, pp. 245–263.
[30] F. Attneave, "Multistability in perception", *Scientific American*, Vol. 225, No. 6, 1971, pp. 62–71.
[31] M. Gardner, "Mathematical games – A Möbius band has a finite thickness, and so is actually a twisted prism", *Scientific American*, Vol. 239, No. 2, 1978, pp. 12–16.
[32] O. Reutersvärd, *Omöjliga figurer* (*Impossible figures*, in Swedish), Leaflet of an exhibition, Galerie LEGER, Malmö, March 1980.
[33] Z. Kulpa, "Oscar Reutersvärd's exploration of impossible lands", in: 150 *Omöjliga Figurer – Oscar Reutersvärd*, exhibition catalogue, Malmö Museum, Malmö, 1981.

Note added in proof

Of significant relevance to the subject of the paper are also the following papers found out recently by the author:

[34] S.W. Draper, "The Penrose triangle and a family of related figures", *Perception*, Vol. 7, 1978, pp. 283–296.
[35] K. Sugihara, "Classification of impossible objects", *Perception*, Vol. 11, 1982, pp. 65–74.
[36] K. Sugihara, "Mathematical structures of line drawings of polyhedrons – toward man-machine communication by means of line drawings", *IEEE Trans. Pattern Analysis and Machine Intelligence*, Vol. PAMI-4, No. 5, 1982, pp. 458–469.