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Constructing the Sunflower Head

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By assigning the Fibonacci angle of 137.507... degrees between any two consecutive individual flowers (florets), and controlling the logarithmic scatter of the floral positions, one of us (Davis) has constructed the sunflower head, botanically known as the capitulum. A mathematical explanation for the configuration seen on the capitulum thus constructed, which simulates that of a natural sunflower head, has been offered by the first author. The formation of the individual florets on the capitulum which eventually causes the emergence of arcs or spirals on it, whose numbers invariably match with the terms of Fibonacci Sequence, can be explained thus: The florets are formed one at a time on the highly compressed stem which flattens out into the disc. The disc gets widened as more and more florets are differentiated, and the older ones move away from the growing point (centralmost region) and the younger ones get distributed around this central point. A flower primordium is differentiated on a side of the stem apex, and the subsequent florets are generated at a fast rate with a constant time-interval between any two consecutive individuals. As the flowers get differentiated, the tip of the meristematic axis rotates so much so that the older florets are seen to move away from the growing point in logarithmic spirals that approximate to an Archimedes' spiral. Moreover, among any two consecutive florets, the younger one starts differentiating from the axis when the older one is at an angle ϕ_1, such that $\phi_1/(2\pi-\phi_1)$ forms the golden ratio (0.618...). This process continues till the genetic material is finished and the flower head is fully programmed. With favorable environmental conditions, the individual flowers expand at a uniform rate over time along with the simultaneous expansion of the disc. It is shown mathematically that the above theory can explain all the properties of a sunflower head whether large or small.

1. INTRODUCTION

The sunflower plant, scientifically known as *Helianthus annuus*, belonging to the family Compositae is important both for its huge, attractive flowerheads as well as for the edible oil that is extracted from the seeds (botanically, fruits). The leaves of this plant are produced one after another (alternate phyllotaxis) in a single spiral, running clockwise or counterclockwise, any two consecutive leaves making an angular deflection of about 137.5°. As the plant matures, the size of the later formed leaves gets reduced when the tip of the main stem flattens out into the flower-bearing disc known as capitulum. At the base of the disc, there are many reduced leaves called "involucral bracts," which also show spiral patterns. On the top of the disc too, numerous highly reduced, closely set bracts are present, each subtending a small flower in its axil. Although these bracts are also produced one after another, the time interval between the formation of any two consecutive bracts (and hence flowers) is considerably reduced. The oldest flowers are distributed at the periphery of the disc, and younger and still younger flowers are met with as one proceeds from the periphery towards the central point, which represents the tip of the compressed main stem. Each of the petal-like structures seen at the periphery of the disc represents a female or sterile flower known as the ray floret. Bordered by the ray florets on the disc are the numerous, less prominent, regular, bisexual flowers called the disc florets. If a perfectly developing capitulum is observed, it will be seen that the flowers start blooming one by one, a bloomed flower appearing at a place approximately 137.5° away from its immediate older one in relation to the central point. In nature such an ideal head is difficult to come across. Thus, the capitulum of the sunflower is a highly compressed stem whose leaves are reduced to small bracts, each supporting a flower in its axil. The development, blooming, and maturity of the individual flowers follow the sequence commencing from the oldest bract and moving upwards (toward the center).

One's admiration for the aesthetic value of sunflower capitulum could be enhanced if one understands the mechanism involved in the arrangement of the individual flowers on it. Spirals or arcs in specific numbers are formed according to certain mathematical laws. Since 1754, several biologists and mathematicians have been attracted by the spiral patterns in nature such as those in the leaf arrangement and arrangement of flowers on the sunflower head and similar other plant organs who gave possible explanations for the same [1–32]. However, none of them could explain fully the different phenomena manifested on the capitulum of Compositae.

Many of the investigators referred to above had realized the importance of some basic properties of the Fibonacci Sequence for explaining the arrangement of leaves on a shoot or the florets on a capitulum which display spiral patterns. The Fibonacci Sequence, named after a 13th century Italian mathematician, Leonardo da Pisa, runs thus: $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233$, etc. A number is obtained by adding the previous two terms. The ratio of any two consecutive Fibonacci Numbers (higher ones) turns out to be $0.618\ldots$ known as the Golden Ratio. It is on account of this peculiar situation that shoots with closely set alternate leaves or the head of a sunflower or other similar organs show spiral mechanism and the number of spirals match a Fibonacci Number.

The configuration seen in the flowerhead shown in Fig. 1 gives the false appearance that the basic pattern is formed by the several arcs. But these arcs are the resultants of a single basic arrangement of the individual flowers. Another interesting property which most workers obviously overlooked is that the individual flowers/seeds are more or less of the same size and shape whether they are formed near the periphery or near the centre of the disc. Considering all these factors, a small head of sunflower having only 234 individual flowers/seeds has been constructed and shown in Fig.

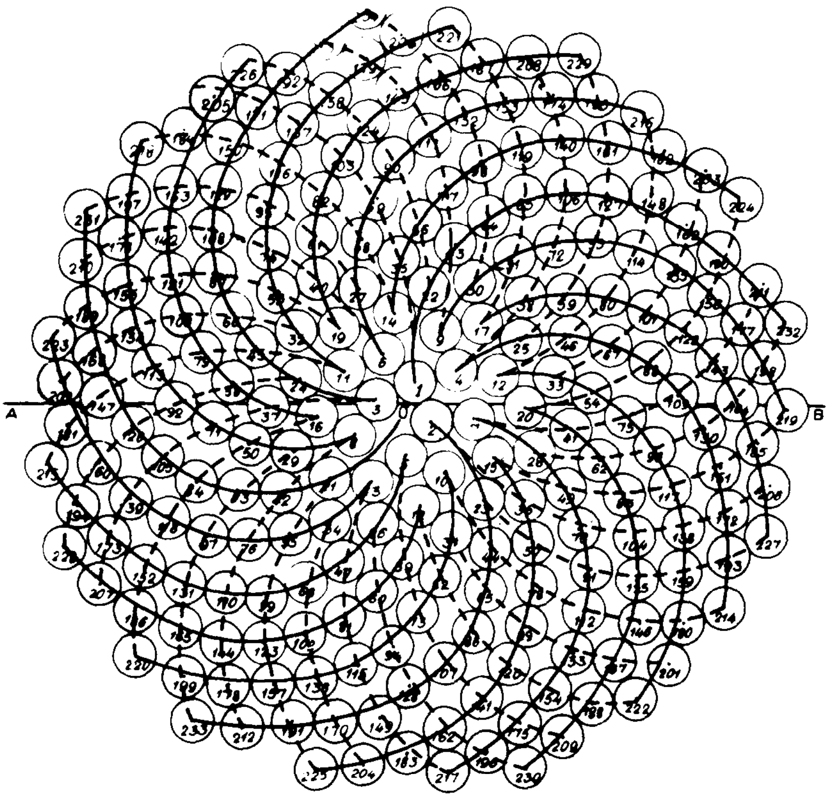

FIG. 1. A schematic head of sunflower showing 34 and 21 spirals.

1. The construction procedure is mentioned under Sec. 2 and a mathematical explanation for the same is given under Sec. 3.

2. CONSTRUCTION

The flowerhead shown in Fig. 1 is constructed in the following manner. A line AB is drawn as the reference axis. Let point 0 on it represent the very tip (growing point) of the shoot that enlarges into the capitulum. A circle with a fixed diameter is drawn touching AB at 0. A second circle of the same diameter is drawn at an angular deflection of 137.507° from the line connecting the center of the first circle and 0. The second circle, which just touches the first one, can be placed either to the left of the first one or to its right. This decides whether the capitulum becomes left-handed or right-handed. In nature the lefts and rights are distributed randomly. In the present case, the second circle is shown on the left of the first. Successive circles of the same diameter have been drawn one after another on the left of the preceding ones at the same angular deflection of 137.507°. None of the subsequent circles overlaps any of their neighbors that lie on the line connecting the center. Although the subsequently formed circles in the picture seem to move away from the center than the early formed ones, a reverse course takes place on the head. That is, flower No. 234 is the oldest, and No. 1 the youngest. At the time when the second flower (circle) is differentiated at the tip of the stem, the first one rotates away from its original position by about 137.5°. Thus, when more and more florets are formed, the already formed ones are pushed towards the periphery with minimum space between them. Some of the earliest formed flowers turn into the ray florets, and the latter ones into disc florets. Care was taken that no circle was nearer to the center than any of the circles already drawn. It can be shown that the configuration in the capitulum does not depend on the size of the circles as long as every circle has the same diameter. Figure 1 exhibits the following property. Start with any circle at the periphery and join the centers of adjoining circles whose numbers differ by 21. Then, the continuous-line radial arcs are obtained. Similarly starting from any circle at the periphery, the circles whose numbers differ by 34 are connected. The dotted-line arcs which move opposite to the 21 arcs are obtained. The two kinds of arcs pass through all the circles.

The overall pattern of Fig. 1 is slightly modified as to bring forth the individual seeds on the capitulum boldly, and the result is seen in Fig. 2. As the radial arcs are drawn just bordering the circles, the space available for a seed is represented by each circle (parallelogram). Thirty four clockwise and 21 counter-clockwise arcs are clearly seen at the periphery. The 34 arcs, each after passing through 4 or 5 seeds, merge with the set of 13 arcs

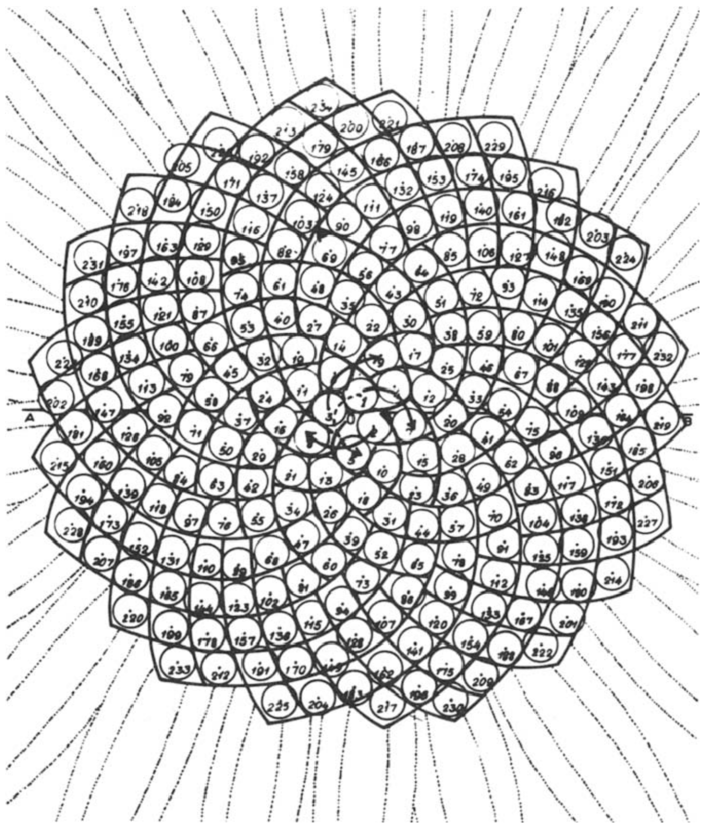

FIG. 2. A reconstructed head of a sunflower.

moving along the same direction but showing a greater degree of curvature. From where the 13 arcs take over from the 34, the original 21 arcs are seen moving opposite to the 13 arcs. Thus, the 34 and 21 arcs seen at the periphery of the head get reduced to 21 and 13. At some level still closer to the center, arcs 13 and 8 are seen and this combination progressively gets reduced to 8 and 5, 5 and 3, and finally to 3 and 2. It is striking that in a natural capitulum one can see a similar configuration (Fig. 3). It is, therefore, emphasized that all the arcs which start from the periphery *do not* reach the center as most earlier workers had erroneously depicted. Richards [22, 23] seems to be the only worker whose illustrations on the phyllotaxis may be considered throwing considerable light on this peculiarity.

It can be inferred from Fig. 2 that in capitulums bearing equal-sized seeds or flowers, the numbers of visible arcs will vary with the diameter of the head, a smaller one showing lower numbers and larger ones higher numbers, all matching with terms in the F. Sequence. The oldest seed/flower of the capitulum in Fig. 2 is just older to 233 members. Two hundred thirty three is a F. Number. Hence a head having 233 to 144 (next lower F. Number) seeds/flowers will show 34 and 21 arcs at the periphery. Another having 89 seeds/flowers will show 21 and 13 spirals, one with 34 seeds will show 13 and 8 spirals, and so on to smaller numbers of arcs as the numbers of individual flowers per head decrease. On the same sunflower plant, the terminal head, which is generally the largest, will show the highest number of arcs at the periphery. But the heads produced on secondary and further orders of branches become smaller and smaller, and they show lesser and lesser numbers of arcs. Another indication for this decrease can be had from the number of ray florets seen on a capitulum. The periphery of the capitulum in Fig. 2 has 21 projections. The flower at each projection develops into a ray floret which increases the attractiveness of the flowerhead, a device to facilitate cross pollination. In a smaller head, the number of projections is reduced to 13, 8, and perhaps 5 and may show an equal number of ray florets. But in nature, considerable variation is observed. However, if data are obtained on a large number of heads, the numbers of ray florets will cluster around some terms of the Fibonacci Sequence. Another striking similarity between this reconstructed picture and a natural capitulum is the following. If any arc is followed from the periphery, the seed at the junction where three or four kinds of arcs meet, is slightly irregular in shape. Such irregular "junction seeds" can be picked out from any large capitulum with developed seeds. At the centralmost part of the head, it is often possible to find very large and highly irregular seeds.

Proceeding from the periphery, one can see the 34 spirals moving clockwise and then joining the 13 spirals which also move clockwise. They are taken over by the 5 spirals that ultimately join the 2 spirals (marked within circles Nos. 1, 3, 5) both moving along the same direction. The spiral numbers synchronizing the alternate terms of the Fibonacci Sequence, i.e., 21, 8, 3, and 1 commencing from the periphery move counterclockwise. This peculiarity of the spirals synchronizing alternate Fibonacci Numbers moving along a common direction has a striking similarity with a basic property of Fibonacci Numbers. The ratio between consecutive F. Numbers swing steadily between the plus and minus sides of the golden ratio (0.618...). The ratio of $1/1$ is 1.0 which is on the plus side, $1/2=0.5$ which is on the minus side; $2/3=0.667$ is on the plus side; $3/5=0.60$ is on the minus side, and so on.

One incorrect approach followed by most earlier investigators is the following. They represent the margin of a capitulum by a circle which is divided into 55 and 89 equal parts (or any two other consecutive F. Numbers). Logarithmic spirals are drawn, all starting from the center of the circle and passing through the 55 points. Similarly, 89 log spirals are drawn (also commencing from the center), which move steeply and opposite the 55 spirals. In the ensuing pattern, the two sets of spirals/arcs cut each other at different points forming "parallelograms." The area of a parallelogram at the periphery turns out to be several times larger than one near the center. This would suggest that the size of the seeds vary considerably depending on the position of the disc where the seed develops. But in nature, the seeds are almost of the *same size* whether they develop from the periphery or at the center of the capitulum (Fig. 3). Other workers who had realized the importance of the Fibonacci angle (137.5°) between successive flowers end with generating log spirals which steadily diverge as they extend from the center. In such diagrams, the parallelograms at different regions of the head would inevitably exhibit wide disparity in their area.

However, these log spirals may hold well enough to represent the growing point of a palm crown or other plants where the various leaves are produced at long time intervals, and therefore, showing a gradation in the size of the leaves. In the sunflower head, such a time interval is very much narrowed down, and there are innumerable units to accommodate. Hence, a spiral with even a small log effect will cause a big difference.

The method of plotting the circles in Figs. 1 and 2 may be explained thus. Assume the various points or centers of parallelograms which are

FIG. 3. A typical compositae capitulum showing ray and disc florets.

obtained by observing the Fibonacci angle between consecutive points in a pattern which show log spirals, are replaced by uniform discs. The margin of disc No. 1 touches 0 on the reference line AB. Disc No. 2 is brought to 0 by the shortest distance and left where it touches disc No. 1 without disturbing the position of disc 1. Similarly, No. 3 is dragged by the shortest track towards 0 and left where it touches the disc nearest to it. Similarly, discs 3,4,5,...234 are brought closer. The required arcs are obtained by connecting the suitable circles. The final configuration (Fig. 2) becomes much different from the original figure.

Also on the lower side of the capitulum where the leaves are modified into involucral bracts, it is possible to trace out different sets of spirals. If Fig. 2 is to represent the pattern of these involucral bracts, the oldest bract will come almost at the center near circle No. 1, and the youngest one to the periphery. As the number of such bracts on any flower-head is much less than that of the flowers on top of the disc, the bracts display arcs matching smaller terms in the Fibonacci series compared to those on the top. Generally a disc showing a maximum of 34 and 21 arcs at the periphery has 21 bracts, and so on, subject to some variation. Like these involucral bracts, the normal leaves themselves display arcs when looked at from the top of a non-flowering shoot. Thus, it is the golden ratio of the angular deflection between any two leaves, bracts or flowers that brings about the radial arcs, their numbers always matching with stages in Fibonacci Sequence.

A sunflower head can also be considered to represent a two-dimensional configuration of the crown of a coconut palm, where the individual seeds would correspond to the different leaves on the crown. The shape of the palm crown as well as the positions of its leaves can be shown to be "best placed" from different mathematical considerations. Efficiency factors can be better explained in a palm crown rather than in a sunflower head because the sunflower disc is a very flattened cone and in order to appreciate the explanation of the efficiency factors, the height of the cone should be larger than what is noticed in a sunflower head. From the mathematical explanation, one can see that the growth principle is the same whether it is in the formation of individual flowers on the capitulum or in the formation of leaves in a palm crown.

3. MATHEMATICAL EXPLANATION

The mathematical equation for a logarithmic spiral in polar coordinates is as follows:

$$r = ke^{a\theta}, \tag{1}$$

where k and a are constants. Since k is only a weighting factor for r,

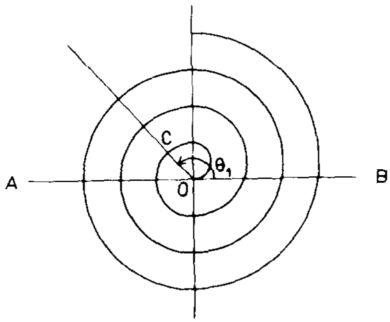

FIG. 4.

without loss of generality we may assume the equation to be of the form

$$r = e^{a\theta}. \tag{2}$$

Figure 4 gives the logarithmic spiral where AB is a reference line and 0 is the origin. Depending upon the value of a, the width between successive spirals increases slowly or rapidly. Consider the case where a is very small. Then we have,

$$r = e^{a\theta} = \left[1 + a\theta + \frac{(a\theta)^2}{2!} + \frac{(a\theta)^3}{3!} + \cdots \right] \approx 1 + a\theta, \tag{3}$$

where "$\approx$" means "approximately equal to," that is, $r - 1 \approx a\theta$. In other words, this logarithmic spiral behaves like an Archimedes' spiral whose equation is of the form,

$$r = b\theta, \tag{4}$$

where b is a constant. Archimedes' spiral has the property that the width between successive spirals remains the same.

Now consider the formation of a sunflower. Let the individual flowers come out at a constant speed of λ units per second along the spiral in Eq. (2) with a very small a. Let the second flower emerge after t_1 units of time. At this stage let the angle be θ_1. That is, the first flower is at the position marked C in Fig. 4 when the second flower starts emerging. If the distance covered by a moving point, such as flower number 1, is denoted by s, then from elementary calculus we have

$$\lambda = \frac{ds}{dt} = \sqrt{r^2 + \left(\frac{dr}{d\theta} \right)^2} \cdot \frac{d\theta}{dt} \tag{5}$$

where, in general ds/dt denotes the derivative of s with respect to t. At time t_1, the distance traveled is λt_1 and thus by integrating Eq. (5) we have,

$$\lambda t_1 = \sqrt{1+a^2}\left[\frac{e^{a\theta_1}-1}{a}\right]. \tag{6}$$

That is,

$$\theta_1 = \frac{1}{a}\log\left\{1+\frac{a\lambda t_1}{\sqrt{1+a^2}}\right\} \tag{7}$$

where the logarithm is taken to the base e. Since λ and a are mutually non-dependent, we may expand the righthand side of (7). That is,

$$\theta_1 = \frac{1}{a}\left\{\frac{a\lambda t_1}{\sqrt{1+a^2}} - \frac{1}{2}\left(\frac{a\lambda t_1}{\sqrt{1+a^2}}\right)^2 + \cdots\right\} \approx \frac{\lambda t_1}{\sqrt{1+a^2}}, \tag{8}$$

(since a is assumed to be very small).

Now when the second flower emerges at position C, the third flower starts and the first flower has travelled another λt_1 units of distance. If the angle which the first flower now makes with AB is denoted by θ_2, then we have,

$$\theta_2 \approx \frac{\lambda t_2}{\sqrt{1+a^2}} = \frac{2\lambda t_1}{\sqrt{1+a^2}} = 2\theta_1 \tag{9}$$

In other words, the angle between successive flowers remains approximately the same as θ_1. In drawing Fig. 1 we have used this property of a logarithmic spiral when the constant a is small. We have taken θ_1 as 137.5°, which is observed in actual sunflower capitulum. In a natural specimen, we know that all the individual florets or seeds are of the same dimension. This observation can be taken into account only if the spiral resembles an Archimedes' spiral where the width between successive spirals remains the same. Thus, it is noticed that the constant a has to be very small. By this process a microscopic flowerhead is formed as in Fig. 5.

If the diameter of the disc of this microscopic flowerhead which is PQ in Fig. 5, is denoted by M, then when environmental conditions are favorable, this M expands at a uniform rate $dM/dt = \mu$ where μ is a constant. Correspondingly, the individual flowers which are denoted by circles in Fig. 5 also expand and finally the disc attains the form as shown in Fig. 1. Now for convenience we will call the last flower as flower number 1 and so on. Then we can get the radial distance of the nth flower and the angle which

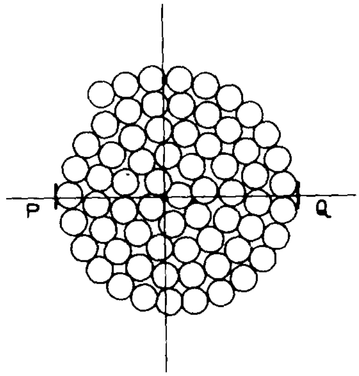

FIG. 5.

nth flower makes with the reference line AB. Let the radial distance of the nth flower be denoted by r_n and the corresponding angle θ_n. Then from (6) we have

$$\lambda t_2 = \frac{\sqrt{1+a^2}}{a}(e^{a\theta_2}-1). \tag{10}$$

But

$$\lambda t_2 = 2\lambda t_1 = 2\frac{\sqrt{1+a^2}}{a}(e^{a\theta_1}-1). \tag{11}$$

From (10) and (11) we have

$$e^{a\theta_2} = 1 + 2(e^{a\theta_1}-1). \tag{12}$$

Proceeding in this way we have

$$e^{a\theta_n} = 1 + n[e^{a\theta_1}-1] \text{ or } \theta_n = \frac{1}{a}\log\{1+n(e^{a\theta_1}-1)\}, \tag{13}$$

But

$$r_n = e^{a\alpha_n} = [1 + n(e^{a\theta_1}-1)]$$

$$= [1 - n + ne^{a\theta_1}]. \tag{14}$$

Equations (13) and (14) give the values of θ_n and r_n in terms of θ_1. We can notice a number of interesting properties from Fig. 1. From Eq. (14) and

Fig. 2 we have

$$\begin{aligned} r_1 - r_{35} &= 34(1 - e^{a\theta_1}), \\ r_{22} - r_{56} &= 34(1 - e^{a\theta_1}), \end{aligned} \quad \text{(along the continuous line arcs),} \qquad (15)$$

$$\begin{aligned} r_{64} - r_{85} &= 21(1 - e^{a\theta_1}), \\ r_{98} - r_{119} &= 21(1 - e^{a\theta_1}), \end{aligned} \quad \text{(along the dotted line arcs).} \qquad (16)$$

In other words, if one continuous line arc is given, then all the other similar arcs can be traced by a simple rotation. Similarly, if one dotted line arc is given, then all other similar arcs can be obtained. In Fig. 2, there are several parallelograms. For example, $r_{43}, r_{64}, r_{77}, r_{98}$ form a parallelogram and further,

$$\frac{r_{64} - r_{43}}{r_{77} - r_{43}} = \frac{21(1 - e^{a\theta_1})}{34(1 - e^{a\theta_1})} = \frac{21}{34}, \qquad \theta_1 \neq 0. \qquad (17)$$

If we consider a parallelogram nearer to the origin of the configuration in Fig. 2, we have for example,

$$\frac{r_9 - r_4}{r_{17} - r_9} = \frac{5}{8}. \qquad (18)$$

It is easy to notice that if we move from the outer perimeter to the origin of the configuration along any radial line, then at different stages one can redraw the two types of arcs in the following sequence: (21,34), (13,21), (8,13), (5,8), (3,5).

This section illustrates that if the individual flowers emerge at a uniform speed at fixed intervals of time along a logarithmic spiral, $r = e^{a\theta}$ with a very small a and with the initial angle of $\theta_1 = 137.5°$, then the exact configuration of a sunflowerhead can be constructed. The results in Eqs. (15)–(18) are due to the assumptions of a basic logarithmic spiral and the numbers 34,21,13,8,5,3,2 from a Fibonacci sequence appear due to $\theta = 137.5°$ because it is well known that the ratios of a successive Fibonacci Sequence, namely, 1/2,2/3,3/5,5/8,5/13,13/21,... approach the golden ratio $(\sqrt{5} - 1)/2$ and if θ_1 is measured in radians and if $\theta_1/(2\pi - \theta_1) = (\sqrt{5} - 1)/2$, then θ_1 is approximately equal to 137.5°.

From the above explanations and theory it is easy to associate the radial arcs seen in sunflowers with the Fibonacci sequence. Now it ramains to be explained why the microscopic flowers are formed along a logarithmic spiral, $r = e^{a\theta}$ with a being very small when the flower is programmed. One plausible explanation is as follows. From the theory of fluid dynamics it is known that if we consider fluid in uniform motion, that is, the same motion is repeated at equal intervals of time, then the law governing the motion is given by

$$f(t_1)f(t_2)=f(t_1+t_2). \tag{19}$$

where t_1 and t_2 are two intervals of time and f is some unknown function. But (19) is the famous Cauchy functional equation in the Theory of Functional Equations, whose unique continuous solution is

$$f(t)=e^{\alpha t} \tag{20}$$

where α is an arbitrary constant. If the fluid is translated into rotation and expansion, then (20) becomes

$$r=e^{a\theta}, \tag{21}$$

where a is an arbitrary constant. This shows that the logarithmic spiral in (21) may be a natural outcome of the supply of genetic material in the form of pulses at constant intervals of time obeying the law of fluid flow as explained in (19).

4. OTHER ILLUSTRATIONS

As other illustrations we can consider a cactus top (Fig. 6) on which an Archimedes' spiral $r=k\theta$ with $k=0.304$ may be superimposed. As explained earlier, a logarithmic spiral $r=e^{a\theta}$ approximates to an Archimedes' spiral except a shift in r, when a is small. In Fig. 6 also one can see that the

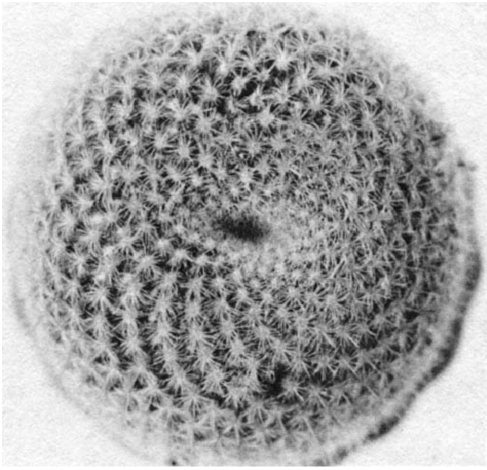

FIG. 6. Top view of a cactus. The various points, where spines cluster together, can be connected by a single Archimedes' spiral.

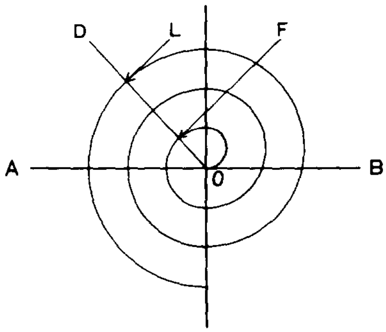

FIG. 7.

individual flowers are formed on a logarithmic spiral with a very small a. Here also one can draw the radial arcs as in Fig. 1, but due to a small number of flowers they will not be very prominent.

5. SOME NUMERICAL CALCULATIONS

The constant appearing in (2) of a logarithmic spiral can be evaluated by measuring the dimensions of actual sunflower seeds. As an illustration we will calculate a for the configuration in Fig. 1. It can be seen that the difference between the radial distances of centers of circles is approximately 1 mm in every seven circles. That is, if the radial distance is measured in mm, then

$$r_1 = e^{a\theta_1}, \tag{22}$$

$$r_n = r_1 + 1 = e^{a\theta_7} = (-6 + 6e^{a\theta_1}) \text{ (from 14)}. \tag{23}$$

That is,

$$a = \frac{1}{\theta_1}\log\left(\frac{7}{6}\right) = \frac{\sqrt{5}+1}{2\pi(\sqrt{5}-1)}\log\left(\frac{7}{6}\right) \tag{24}$$

approximately, since $\dfrac{\theta_1}{2\pi - \theta_1} = \dfrac{\sqrt{5}-1}{2}$.

6. ARRANGEMENT OF FLOWERS ON RADIAL LINES

Occurrence of individual flowers in a capitulum in the form of 5 or 8 or m radial lines can be explained easily. This depends upon the initial angle θ_1. Consider Fig. 7. When the individual flowers are forming on a logarithmic spiral, we have from Eq. (13),

$$\theta_n = \frac{1}{a}\log[1 + n(e^{a\theta_1} - 1)] \tag{25}$$

or

$$e^{a\theta_n} = (n-1)(e^{a\theta_1} - 1). \tag{26}$$

Suppose that the nth flower coincides with the first flower in the sense that both make the same angle with the radial line OD in Fig. 7. Here F and L denote the first and nth flowers. When this happens, let the number of coils which the logarithmic spiral makes be $m+1$, then,

$$\theta_n = 2\pi m + \theta_1. \tag{27}$$

In Fig. 7 we have 3 coils seen cutting OD. If the first flower F makes an angle θ_1 with OD, where AB is the reference line, then L makes the angle $2(2\pi)+\theta_1$. Thus we have,

$$e^{a(2m\pi+\theta_1)} = (n-1)(e^{a\theta_1}-1)$$

or

$$\theta_1 = \frac{1}{a}\log\left[1+\frac{1}{n-1}e^{2am\pi}\right]. \tag{28}$$

If θ_1 is determined by (28) for a preassigned n and m, then we will have flowers arranged in radial lines, the number of lines depending upon m and n. In all such cases, the generation or the programming of the flowers is governed by the same principle of a basic logarithmic spiral with a small constant a eventhough different patterns or different types of radial arcs are seen.

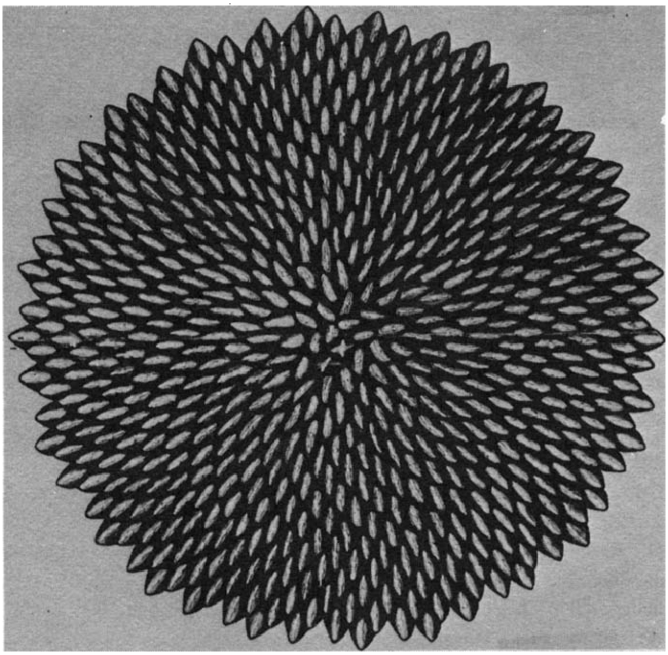

FIG. 8. Reconstructed head of a sunflower.

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REFERENCES

1 D. Bergamini, *Mathematics*, Life Science Library, (1965).

2 C. Bonnet, *Recherches sur l'Usage des Feuilles dans les Plants*, Gottingen and Leiden (1754).

3 A. Braun, Vergleichende Undersuchung über die Ordnung der Schuppen an den Tannenzapfen. *Nova Acta Acad. Car. Leop.* **15**, 195–401 (1831).

4 A. H. Church, *On the Relation of Phyllotaxis to Mechanical Laws*, London (1904).

5 A. H. Church, *On the Interpretation of Phenomena of Phyllotaxis*, New York (1920).

6 S. Colman and C. A. Coan, *Proportional Form*, New York (1920).

7 H. S. M. Coxeter, *Introduction to Geometry*, New York (1961).

8 H. S. M. Coxeter, The golden section, phyllotaxis and Wythoff's game. *Scripta Math.* **19**, 135–143 (1953).

9 T. A. Davis, Fibonacci Numbers for palm foliar spirals. *Acta Botanica Neerlandica* **19**, 249–256 (1970).

10 T. A. Davis, Why Fibonacci Sequence for palm leaf spirals? *Fibonacci Quart.* **9**, 237–244 (1971).

11 T. A. Davis and T. K. Bose, Fibonacci System in aroids. *Fibonacci Quart.* **9**, 253–263 (1971).

12 C. De Candolle, *Considerations sur l'Etude de la Phyllotaxis*, Geneva (1881).

13 H. Wiesbaden Ducker and H. Leith, Mathematische probleme in de biologie. II. *PM2*, **H10**, 257–261 (1960).

14 Frank Land, *The Language of Mathematics*, London (1960).

15 M. Gardner, The multiple fascinations of the Fibonacci Sequence. *Sci. Amer.* **220**, 116–120 (1969).

16 K. Goebel, *Organographie der Pflanzen*, Fischer, Jena (1928).

17 W. L. Kilmer, On growing pine cones and other Fibonacci fruits. McCulloch's Localized Algorithm. *Math. Biosci.* **11**, 53–57 (1971).

18 E. E. Krammer, *The Main Stream of Mathematics*, New York (1955).

19 E. E. Leppik, Phyllotaxis, anthotaxis, semataxis. *Acta Biotheor.* **14**, 1–28 (1961).

20 E. E. Leppik, Evolutionary differentiation of the flower head of the Compositae. II. *Annales Botanici Fennici* **7**, 325–352 (1970).

21 L. Plantefol, La theorie des helices foliaires multiples, *Fundaments d'une Theorie Phyllotaxique Nonvelle*, Masson, Paris (1948).

22 F. J. Richards, The geometry of phyllotaxis and its organ, *Soc. Exper. Biol. Symp.* **II**, 217–245 (1948).

23 F. J. Richards, Phyllotaxis: its quantitative expression and relation to growth in the apex. *Phil. Trans. Roy. Soc. (London)* **B235**, 509–564 (1950).

24 C. F. Schimper, Geometrische Anordung der umeine Achse peripherische Blattgebilde, *Verhandl. Schweiz. Ges.* **1836**, 113–117 (1836).

25 S. Schwendener, *Mechanische Theorie der Blattstellung* Engelman, Leipzig (1878).

26 E. W. Sinnott, *Plant Morphogenesis*, New York (1960).

27 R. Snow, Problems of phyllotaxis and leaf determination. *Endeavour*, **14**, 190–199 (1955).

28 Mary Snow and R. Snow, The interpretation of phyllotaxis, *Biol. Rev.*, **9**, 132–137 (1934).

29 D. W. Thompson, *On Growth and Form*, Cambridge (1917).

30 C. W. Wardlaw, Leaf formation and phyllotaxis in *Dryopteris aristata* Druce. *Ann. Bot.*, **13**, 163–198 (1949).

31 A. Weisse, Die Zahl der Randblüten an Compositen—Kopfchen, *Jahrb. Wiss. Bot.*, **30**, 453–483 (1897).

32 C. Wright, On the uses and origin of arrangements of leaves in plants. *Mem. Amer. Acad. Arts. Sci.*, **9**, 379–415 (1873).