tensor of the first rank, with respect to any nonsingular coordinate transformation, which behaves in every respect like an ordinary polar vector. If **u** and **v** are polar vectors, then it is useful to call $\mathbf{u}\times\mathbf{v}$ an axial vector in three dimensions on the basis of its *formation* and not on its transformation.

A final remark is necessary in this note with respect to the conventional view that an axial vector is related to an antisymmetric tensor of the second rank. The definition in Eq. (16) leads to a revision of this view. We find, rather, that the *vector operator* $\mathbf{u}\times$ is to be the mathematical entity which is uniquely related to an antisymmetric tensor of the second rank. We are referring to the operation on vectors. This can be put into mathematical form quite generally. To do this, let us drop the restriction that $\mathbf{a}^1$, $\mathbf{a}^2$, $\mathbf{a}^3$ and $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ are to be unit vectors and let them be noncoplanar reciprocal affine reference sets, namely:

$$\mathbf{a}_r\cdot\mathbf{a}^s=\delta_r{}^s=\begin{cases}1, & r=s\\ 0, & r\neq s,\end{cases} \qquad (19)$$

where Eqs. (17) and (18) still apply. We have the following:

Theorem. If **A** is an antisymmetric tensor, it can be written in one and only one way as the vector operator $\mathbf{a}\times$ where **a** is a uniquely determinable vector if **A** is known, as follows:

$$\mathbf{a}\equiv\tfrac{1}{2}\mathbf{a}^\sigma\times\mathbf{A}\cdot\mathbf{a}_\sigma, \qquad (20)$$

summation on $\sigma=1, 2, 3$.

Proof. The tensor **A** is completely characterized by its application to the three noncoplanar vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$. Let us assume $\mathbf{A}\cdot\mathbf{a}_r=\mathbf{b}_r$, $r=1, 2, 3$. We need only show that

$$\begin{aligned}\mathbf{a}\times\mathbf{a}_r&=\tfrac{1}{2}(\mathbf{a}^\sigma\times\mathbf{A}\cdot\mathbf{a}_\sigma)\times\mathbf{a}_r=\tfrac{1}{2}(\mathbf{a}^\sigma\times\mathbf{b}_\sigma)\times\mathbf{a}_r\\&=\tfrac{1}{2}[(\mathbf{a}^\sigma\cdot\mathbf{a}_r)\mathbf{b}_\sigma-(\mathbf{a}_r\cdot\mathbf{b}_\sigma)\mathbf{a}^\sigma]\\&=\tfrac{1}{2}[\delta_r{}^\sigma\mathbf{b}_\sigma-(\mathbf{a}_r\cdot\mathbf{A}\cdot\mathbf{a}_\sigma)\mathbf{a}^\sigma]=\tfrac{1}{2}[\mathbf{b}_r+(\mathbf{a}_\sigma\cdot\mathbf{b}_r)\mathbf{a}^\sigma]\\&=\tfrac{1}{2}[\mathbf{b}_r+\mathbf{b}_r]=\mathbf{b}_r,\quad r=1, 2, 3,\end{aligned}$$

where we have invoked the antisymmetry of **A** by taking

$$-\mathbf{a}_r\cdot\mathbf{A}\cdot\mathbf{a}_\sigma=+\mathbf{a}_\sigma\cdot\mathbf{A}\cdot\mathbf{a}_r.$$

From the point of view favored here, the scalar triple product

$$\mathbf{u}\cdot\mathbf{v}\times\mathbf{w}=a^{\rho\sigma\tau}u_\rho v_\sigma w_\tau, \qquad (21)$$

is an absolute scalar and not a pseudoscalar. In this way, we give to three dimensional vector analysis a uniform invariant tensor character. This is more attractive than that generally espoused in the literature of mathematical physics where the discussion of two kinds of vectors based on differences in transformation properties is the result of an incomplete definition of the vector product. From the physical point of view, we note that axial vectors are not directly observable concepts but are either inferred from a calculation or an observable effect.

In summary, there is no gainsaying the fact that right-handed and left-handed reference systems are different mathematically and physically. The mathematical difference is expressed by the value of the Jacobian which is -1 and the physical difference is that the systems cannot be superposed by translation and rotation. The definitions in Eqs. (1) and (2) express these differences. The vector algebra which follows from these definitions is a self-consistent invariant tensor formulation.

[1] L. Landau and E. Lifschitz, *Classical Theory of Fields* (Addison-Wesley Publ. Co., Reading, Mass., 1951), pp. 19–20.
[2] P. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Company, Inc., New York, 1953), Part I, p. 10.

Two-Circle Roller

A. T. Stewart
University of North Carolina at Chapel Hill

There is a child's building toy which makes use of plastic circles notched around their edges as illustrated in Fig. 1. After these pieces were spread haphazardly

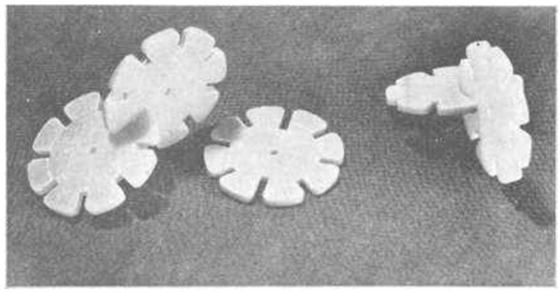

Fig. 1. Plastic circles of a child's building toy.

throughout our house we seldom saw at any one time more than two pieces fitted together, the connected circles shown in the figure. These pairs of circles have a very striking property which brought them to my attention. They roll easily on a slight incline showing little tendency to find a position of stationary equilibrium. (In addition it should be said that the motion is a curious wobble quite amusing to watch!) The question arises naturally: what is the vertical motion of the center of mass during the rolling?

This note points out that a pair of thin circles joined at right angles with centers separated by a distance d, equal to $(r)^{\frac{1}{2}}$, the radius, does indeed *roll with the height of the center of mass remaining constant.* Furthermore, for considerable departure from the condition $d=r(2)^{\frac{1}{2}}$, the vertical motion of the center of mass is very small. This result is shown in Fig. 2. The purpose of this note is in

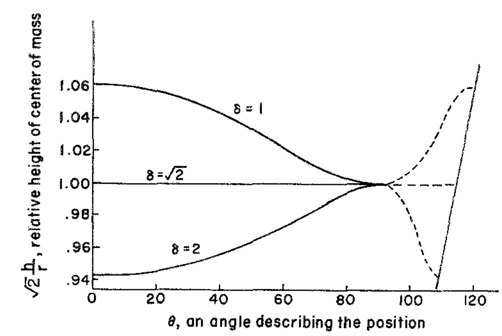

Fig. 2. The variation of height of center of mass with the angle describing the rolling motion. See text for definition of symbols.

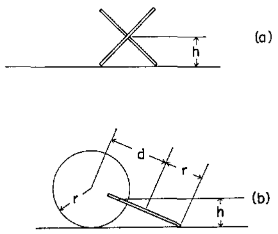

FIG. 3. The two positions of symmetry.

part to point out this interesting behavior but more especially to seek a simple proof of it. If any know of a simple solution I would like to hear of it.

There are two symmetric positions sketched in Fig. 3. In the position of Fig. 3(a) it is obvious that the height of the center of mass, h, is

$$h = r/(2)^{\frac{1}{2}}.$$

In the symmetric position drawn in Fig. 3(b) it is easy to show that the center of mass is at height

$$h = r[(1+\tfrac{1}{2}\delta)/(1+\delta)],$$

where $\delta = d/r$. The heights in these two positions are equal if

$$1/(2)^{\frac{1}{2}} = (1+\tfrac{1}{2}\delta)/(1+\delta),$$

which yields the condition

$$\delta = (2)^{\frac{1}{2}} = d/r.$$

No other position can be analyzed so easily.

For an arbitrary position, a lengthy analysis shows that the height of the center of mass is given by

$$h = [r/(2)^{\frac{1}{2}}]\{(1+\tfrac{1}{2}\delta C)/[1+\delta C+\tfrac{1}{2}(\delta^2-1)C^2]^{\frac{1}{2}}\},$$

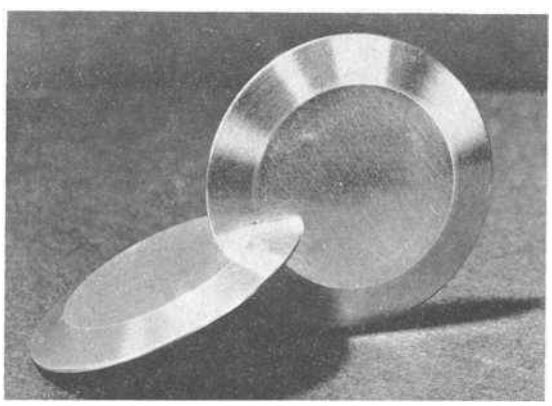

FIG. 4. A metal two-circle roller which performs very well.

where C is the cosine of the angle between the line joining the centers of the two circles and one of the radius vectors which contacts the supporting table. (In Fig. 3(a) this angle is 90° for both circles. In Fig. 3(b) the angle is 0° for the lower circle and greater than 90° for the vertical circle.) The function $(2)^{\frac{1}{2}}h/r$ is plotted against angle in Fig. 2. It can be seen that the above expression reduces to $h = r/(2)^{\frac{1}{2}}$ for $\delta = (2)^{\frac{1}{2}}$ so that h is invariant with rolling. It would be interesting to know of some more direct proof of this simple property.

Demonstration of Single Crystal Making

RONALD A. AZIZ
University of Waterloo, Waterloo, Ontario, Canada

A SIMPLE, effective classroom demonstration of the production of a single crystal is described. A mould is fabricated from a piece of Pyrex tubing by drawing it to a point at one end. The tip is then ground flat until a tiny hole (0.2 to 0.3 mm in diameter) appears (Fig. 1).

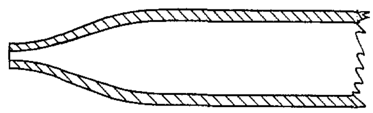

FIG. 1. Cross-section of Pyrex glass mould.

Using an acetylene torch, tin (99.999% pure) is melted within a high-purity graphite crucible. The oxides which appear remain principally on the surface of the molten tin. The tin is then poured into the preheated mould, and the torch is applied to the system to ensure that the metal is still molten. The mould is then lowered until the tip touches an oil bath. Heat is transferred via the metal in the channel in the tip to the oil bath, and a solid–liquid interface moves slowly up the tube. A sheet of aluminum foil may be used to insulate the tube, itself an insulator, to avoid premature chilling of the molt. To remove the cylindrical tin crystal, the system is placed, once cooled to room temperature, into a liquid nitrogen bath. The tin sample contracts away from the tube and can be shaken free from the tube. The structure of the sample can be revealed by placing it for a few seconds into an etch of the following composition: 9 parts HCl, 3 parts HNO_3, 2 parts HF, and 5 parts H_2O (distilled). The ends can be squared off with a spark cutter. While a single crystal often results whenever high purity tin is used, it is found that the "mortality rate" is reduced when this method is employed. There may be some contamination of the sample due to absorption of impurities from the Pyrex.