lens properties which may not have been noted previously. They make possible a quick and comprehensive comparison of the characteristics of different lenses or of lenses of the same type with different di-

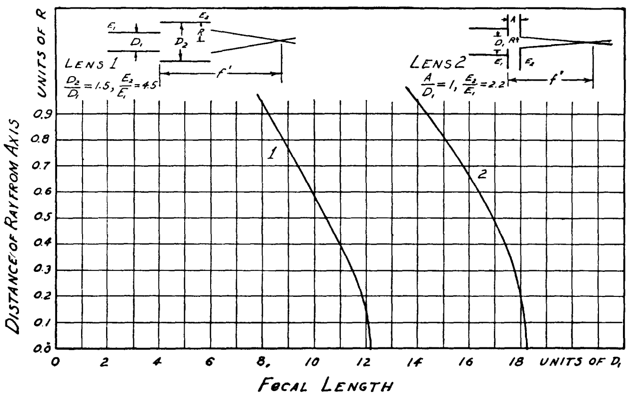


Fig. 8—Spherical aberration of lenses: Variation of focal distance with aperture.

mensions. Such a comparison is shown in Fig. 7 which shows the effect of changing the ratio of aperture diameter to aperture spacing in an aperture lens. In this figure, object distance and image distance are measured from the first aperture, $A$ is the aperture diameter, and $D$ is the axial distance between apertures.

*C. Aberration Characteristics*

Some typical aberration curves as determined graphically from the screen patterns are shown in Fig. 8. These show the decrease in focal distance as the ray separation from the axis is increased. Such curves are about the same for all lenses. These curves are nearly universal in that the reduction in focal distance is approximately a percentage function of the focal distance itself. A sample curve showing the spread of spot pro-

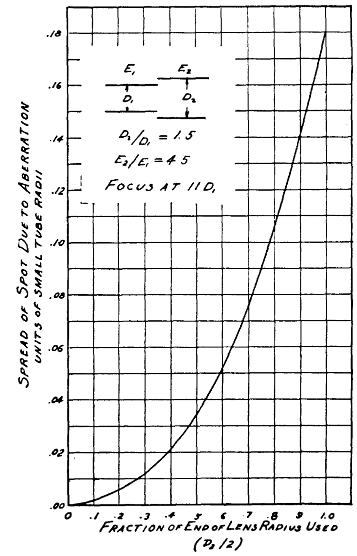


Fig. 9—Aberration curve: Variation of minimum spot size with aperture.

duced by aberration is shown in Fig. 9. This is also nearly a universal curve.

## IV. Conclusions

The methods proposed here are improvements on previously proposed methods from the standpoint of simplicity, ease of execution, and accuracy of results. The new representation of lens characteristics tells the whole story of the lens at a glance. The experimental method permits simultaneous determination of focal characteristics and aberration properties.

# A More Symmetrical Fourier Analysis Applied to Transmission Problems*

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***Summary***—*The Fourier identity is here expressed in a more symmetrical form which leads to certain analogies between the function of the original variable and its transform. Also it permits a function of time, for example, to be analyzed into two independent sets of sinusoidal components, one of which is represented in terms of positive frequencies, and the other of negative. The steady-state treatment of transmission problems in terms of this analysis is similar to the familiar ones and may be carried out either in terms of real quantities or of complex exponentials. In the transient treatment, use is made of the analogies referred to above, and their relation to the method of "paired echoes" is discussed. A restatement is made of the condition which is known to be necessary in order that a given steady-state characteristic may represent a passive or stable active system (actual or ideal). A particular necessary condition is deduced from this as an illustration.*

A NEW formulation of the Fourier integral identity is derived and compared with the familiar ones and its properties are discussed. The application of the resulting analysis to transmission problems, steady-state and transient, follows.

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## Mathematical Relations

*Comparison of Alternative Forms*

The Fourier integral identity may be written in the form

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega\, \psi(\omega) \operatorname{cas} \omega t, \tag{1}$$

$$\psi(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\, f(t) \operatorname{cas} \omega t, \tag{2}$$

where

$$\operatorname{cas} x = \cos x + \sin x,$$

is an abbreviation for cosine and sine. This is to be compared, from the standpoint of symmetry, with the more usual forms,

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\alpha\, f(\alpha) \cos \omega(t - \alpha), \tag{3}$$

or its equivalent,

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega [A(\omega) \cos \omega t + B(\omega) \sin \omega t], \quad (4)$$

$$A(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\, f(t) \cos \omega t, \quad (5)$$

$$B(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\, f(t) \sin \omega t, \quad (6)$$

and

$$f(t) = \frac{}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega\, C(\omega) \exp (i\omega t), \quad (7)$$

$$C(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\, f(t) \exp (- i\omega t).$$

To derive (1) and (2), we write

$$\psi(\omega) = A(\omega) + B(\omega).$$

Then (2) follows from (5) and (6). Since $A(\omega)$ is an even function of $\omega$ and $B(\omega)$, an odd,

$$0 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega [A(\omega) \sin \omega t + B(\omega) \cos \omega t].$$

If we add this to the right member of (4) it reduces to (1).

Equations (3) to (6) are similar to (1) and (2) in that, when $f(t)$ is real, all the other quantities are also real. They differ in that the variables $t$ and $\omega$ enter the equations symmetrically in the latter and not in the former. Equations (7) and (8) resemble (1) and (2) more closely in form. They differ in that the symmetry of (7) and (8) is marred by the difference in sign of the two exponents. Also when $f(t)$ is real, $C(\omega)$ is complex, and vice versa.

We may then set up the following expressions for the even and odd components of $f(t)$ and $\psi(\omega)$:

$$\begin{aligned} f_e(t) &= \tfrac{1}{2}[f(t) + f(-t)], \quad (t > 0), \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega\, \psi_e(\omega) \cos \omega t = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega\, \psi(\omega) \cos \omega t, \end{aligned} \quad (9)$$

$$\begin{aligned} f_0(t) &= \tfrac{1}{2}[f(t) - f(-t)], \quad (t > 0), \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega\, \psi_0(\omega) \sin \omega t = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega\, \psi(\omega) \sin \omega t, \end{aligned} \quad (10)$$

$$\begin{aligned} \psi_e(\omega) &= \tfrac{1}{2}[\psi(\omega) + \psi(-\omega)], \quad (\omega > 0), \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\, f_e(t) \cos \omega t = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\, f(t) \cos \omega t, \end{aligned} \quad (11)$$

$$\begin{aligned} \psi_0(\omega) &= \tfrac{1}{2}[\psi(\omega) - \psi(-\omega)], \quad (\omega > 0), \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\, f_0(t) \sin \omega t = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\, f(t) \sin \omega t. \end{aligned} \quad (12)$$

Equations (3) to (6) differ from (1) and (2) also with respect to negative values of $\omega$. Since the first integrand in (3) is an even function of $\omega$, the component of $f(t)$ corresponding to integration over negative values of $\omega$ is identical with that over positive. If then we regard (3) as an analysis of $f(t)$ into sinusoidal components, we may say that one half of the value of the function is represented by components for which $\omega$ is positive and one half by those for which it is negative. In (1) however we note that

$$\begin{aligned} \operatorname{cas} (-\omega t) &= \sqrt{2} \cos \left( -\omega t - \frac{\pi}{4} \right), \\ &= \sqrt{2} \cos \left( \omega t + \frac{\pi}{4} \right), \end{aligned} \quad (13)$$

and

$$\operatorname{cas} (\omega t) = \sqrt{2} \sin \left( \omega t + \frac{\pi}{4} \right).$$

We may, therefore, say that a pair of equal positive and negative values of $\omega$ in (1) correspond to a pair of components which vary as the sine and cosine of the same angle. Thus (5) and (6) represent a resolution into sine and cosine components each of which is further resolved into components corresponding to $\omega$ and $-\omega$, whereas in (1) these two resolutions are accomplished together.

This difference gives rise to a corresponding one in the functions of $\omega$ by which a given function of $t$ may be represented. Equation (4) suggests that use is made of two functions, $A(\omega)$ and $B(\omega)$, each defined for both positive and negative values of $\omega$. However, in view of their evenness and oddness, they are completely determined by their values over either range alone. In (1), on the other hand, we have a single function $\psi(\omega)$, the value of which for $-\omega$ is independent of that for $\omega$; and so it must be defined over the entire range of $\omega$.

### *Analogous Functions of Time and Frequency*

The symmetry of (1) and (2) makes possible some analogies between functions of $t$ and $\omega$. The discussion of these may be simplified, without loss of generality, if we identify $t$ with time and $\omega$ with angular frequency. It will be further simplified if we replace $\omega$ by $2\pi$ times the cyclic frequency $\nu$, writing (1) and (2) as

$$f(t) = \int_{-\infty}^{\infty} d\nu\, \Phi(\nu) \operatorname{cas} 2\pi\nu t, \quad (14)$$

$$\Phi(\nu) = \int_{-\infty}^{\infty} dt\, f(t) \operatorname{cas} 2\pi\nu t, \quad (15)$$

where

$$\Phi(\nu) = \sqrt{2\pi}\, \psi(2\pi\nu) = \sqrt{2\pi}\, \psi(\omega).$$

We have interpreted equations such as (1) and (4) as representing a resolution of $f(t)$ into sinusoidal components. We may also interpret (2), (5), and (6) as representing the resolution of their respective functions of frequency into components which vary sinusoidally with frequency. For example in (5) (as modified), the component corresponding to a particular instant $t_1$ has the form $dt\, f(t_1) \cos 2\pi t_1 \nu$, as shown in Fig. 1. This value of $t$ determines the amplitude, $dt\ f(t_1)$, and is itself equal to the number of cycles per unit frequency range. Its reciprocal $1/t_1$ is the frequency range occupied by one cycle of the sinusoid. This role of $t_1$ is the analog of that of the frequency $\nu_1$ of a single frequency component of $f(t)$ as given by (4). It seems logical

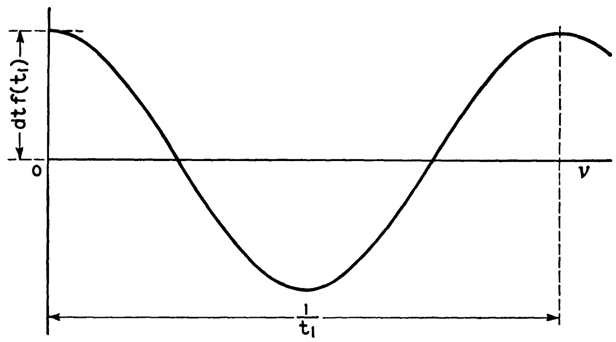


Fig. 1

therefore to refer to one of these sinusoids on the frequency scale as a single instant component of $\phi(\nu)$ and to the corresponding instant as the instant of the component.

The resolution of $\phi(\nu)$ may follow either the familiar method, as in the foregoing example, or the alternative one described above. It would seem that our intuitions would be best satisfied by that resolution in which the values of $f(t_1)$ and $f(-t_1)$ best maintain their separate identities in the components of $\phi(\nu)$. From this standpoint the newer analysis is preferable. In (2) the components corresponding to instants $t_1$ and $-t_1$ constitute a pair of sine and cosine components as in Fig. 2. The corresponding pair in (5) and (6) are found by compounding the components of instants $t_1$ and $-t_1$ for the sine and cosine separately. The amplitudes of the resultants are proportional to $f(t_1)-f(-t_1)$ and $f(t_1)+f(-t_1)$, respectively. Here the identities of the positive and negative instants are pretty well lost.

If we apply the Fourier analysis to a function of time of the form

$$f_1(t) = A \text{ cas } 2\pi\nu_1 t,$$

the resulting function $\Phi_1(\nu)$ is zero except at $\nu_1$ where it has an infinite value such that

$$\Phi_1(\nu_1)d\nu = A.$$

Thus the transform of a cas function of time of finite amplitude is a finite pulse of infinite height and infinitesimal length. When the function $f(t)$ comprises an infinitude of cas components, as indicated in (14), the amplitude of each component is infinitesimally small. For example, the component of frequency $\nu_1$, say, has an amplitude of $\Phi(\nu_1)d\nu$. The transform of this component then is an infinitesimal pulse of finite height $\Phi(\nu_1)$, and length $d\nu$ located at $\nu_1$. Suppose that we transform in the same way all of the other cas components of $f(t)$. The result is a succession of pulses spaced at intervals of $d\nu$, each of a height equal to the corresponding value of $\Phi(\nu)$. The same succession of pulses would result from the well-known resolution of $\Phi(\nu)$ into a succession of infinitesimal pulses of length $d\nu$, each of a finite height, given by the average value of $\Phi(\nu)$ over the particular interval $d\nu$. Also, from sym-

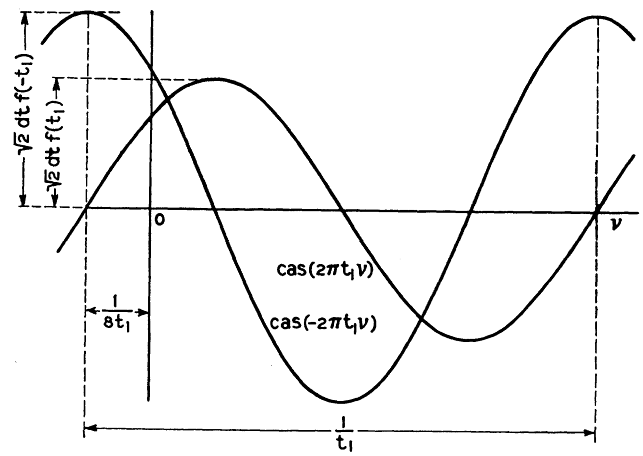


Fig. 2

metry, if we analyze $f(t)$ into pulses $f(t)dt$, and transform these individually we get the same sinusoidal components of $\Phi(\nu)$ that we do if we transform $f(t)$ into $\Phi(\nu)$ and analyze it into sinusoidal components. In general, the amplitude of a sinusoidal component of a function is equal to the magnitude of the corresponding pulse of its transform.

Close approximations to sinusoidal components of a time function are in common use. More rarely experimental use is made of an approximate pulse in the form of a current of finite duration, the magnitude of which is made to vary inversely as the duration as the latter is decreased. The transform of such a wave approximates to a sinusoidal $\Phi$ function of finite amplitude. If also the pulse is an even function of time and occurs at the instant zero, the corresponding $\Phi$ function approaches a uniform finite value for all frequencies.

By analogy with the Fourier series, it is obvious that $\Phi(\nu)$ can be represented over a limited frequency range, by an infinite series of finite, single-instant cas components which are finitely spaced. These correspond to a series of equally spaced finite pulses on the time scale, each of infinite height and infinitesimal duration. The finite time interval separating the pulses is the reciprocal of the range of frequency over which the function of frequency is to be represented. These pulses will be distributed over the entire time scale. If the Fourier integral analysis be applied to this sequence of pulses, the resulting function of frequency will repeat itself on the frequency scale, the interval of repetition being equal to that over which the original function was to be represented.

As an example of this relationship may be mentioned a property of telegraph signals derived by

Nyquist.[1] He assumed that in a synchronous telegraph system employing any number of elements, the duration of each signal pulse is made small compared with the signal interval. He then showed that the spectrum of the over-all disturbance repeats itself on the frequency scale, except for a factor dependent on the form of the pulse. Presumably if the duration of the pulse is sufficiently decreased, the effect of its form can be made negligible. With this assumption, the result becomes a special case of the analog of the Fourier series relation.

## Application to Transmission Problems

In applying the foregoing to transmission problems it will be assumed that the systems under consideration are linear with constants which do not vary with time. They will be assumed to be either passive or stable active systems, except where unstable active systems are specifically mentioned.

### *Steady-State Transmission in Terms of $\psi$ Components*

By the steady-state characteristic of such a system we mean a description, in terms of functions of frequency, of how it transmits sinusoidal waves of various frequencies. This amounts to a statement, for each frequency, of the relation between some two waves associated with the system. If they are the current and voltage at the same point, the relation takes the form of an impedance, consisting of a resistance and a reactance. As the relation between any other pair of waves may be expressed in similar form, there will be no loss of generality if we carry on the discussion in terms of an impedance.

The resistance gives the ratio, to the amplitude of the current, of the amplitude of that component of the voltage which is in phase with the current. The reactance gives a similar ratio for the component in quadrature. When the familiar analysis is used, it is convenient in computing the impedance to choose the current as a sine wave of unit amplitude. The amplitudes of the computed sine and cosine components of the voltage then give the resistance and reactance.

If we wish to carry out the computation in terms of the new analysis, we again assume that the current is represented by one of the quadrature components, in this case that of positive frequency $\psi(\omega_1)$ cas $\omega_1 t$, $(\omega_1 > 0)$. If we analyze this into its sine and cosine components, compute the accompanying voltages in the familiar way, and resolve their resultant into cas components of positive and negative frequency, we get

$$\psi(\omega_1)[R(\omega_1) \operatorname{cas} \omega_1 t + X(\omega_1) \operatorname{cas} (-\omega_1 t)]. \tag{18}$$

Obviously, if we make $\psi(\omega_1)$ unity, the resistance is given by the magnitude of the voltage component of positive frequency and the reactance by that of negative frequency.

[1] H. Nyquist, "Certain topics in telegraph transmission theory," *Trans. A.I.E.E.* (*Elec. Eng.*, April, 1928), vol. 47, pp. 617–644; 1928.

More often perhaps, we know the impedance and wish to compute the voltage which accompanies a given current. If the current is sinusoidal we resolve it into cas components of positive and negative frequency. The voltage accompanying the component of frequency $\omega_1$ is given by (18). That accompanying the component of frequency $-\omega_1$ is obtained by reversing the sign of $\omega_1$ in (18). It is

$$\psi(-\omega_1)[R(-\omega_1) \operatorname{cas} (-\omega_1 t) + X(-\omega_1) \operatorname{cas} \omega_1 t], \tag{18'}$$

where

$$\begin{aligned} R(-\omega_1) &= R(\omega_1), \\ X(-\omega_1) &= -X(\omega_1), \end{aligned} \qquad (\omega_1 > 0).$$

From these results it follows that if the current is a transient which can be represented by

$$I = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega\, \psi(\omega) \operatorname{cas} \omega t, \tag{19}$$

the voltage is

$$E = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega\, \psi(\omega) [R(\omega) \operatorname{cas} \omega t + X(\omega) \operatorname{cas} (-\omega t)], \tag{20}$$

provided $R(\omega)$ and $X(\omega)$ satisfy certain well-known conditions.

The new analysis lends itself to the use of complex algebra in a manner exactly analogous to that of the familiar analysis. In the familiar case we carry out the operations in terms of exp $(i\omega t)$, the real part of which is cos $\omega t$. When this is multiplied by the impedance $R(\omega) + iX(\omega)$, the real part of the product gives the voltage. In the new analysis, we recognize that the sum of the real and imaginary parts of exp $(i\omega t)$ is cas $\omega t$. Hence if we multiply exp $(i\omega t)$ by the complex impedance as before, the sum of the real and imaginary parts of the product gives the voltage. We may then write (19) and (20) in the form

$$I = \text{real} + \text{imaginary part of } \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega\, \psi(\omega) \exp(i\omega t),$$

$$E = \text{real} + \text{imaginary part of } \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega\, Z(\omega)\psi(\omega) \exp(i\omega t),$$

where $\psi(\omega)$ is real and $Z(\omega)$ is the complex impedance. An alternative method is based on the relation

$$(1 + i) \exp(i\omega t) = \operatorname{cas}(-\omega t) + i \operatorname{cas}(\omega t).$$

If the real part of this represents the current, then

$$\text{real part of } (1 + i) Z \exp(i\omega t) = R \operatorname{cas}(-\omega t) - X \operatorname{cas} \omega t,$$

which from (18′) is the voltage. Here the cas component of negative frequency has a role similar to that of the cosine component and that of positive frequency to that of the sine, which is consistent with (13). It is also consistent with the fact that it is the sine and the positive-frequency components of current which are accompanied by voltage components which are equal to the resistance and reactance, respectively. For the cosine and negative-frequency components the quadrature component of the voltage is equal to minus the reactance.

*Systems Characteristics as Spectra of Transients*

It is well known that under suitable conditions the functions of frequency which give the steady-state characteristic of a system are identical with those which describe the transient which accompanies excitation by an ideal pulse at time zero.[2] Such a transient may be analyzed into components of positive and negative frequency, the magnitudes of which are expressed in terms of the system characteristics. The resulting function of frequency bears a more symmetrical relation to the time function representing the transient than do the familiar frequency functions. This symmetry is found to be helpful in establishing relations between the steady-state and transient characteristics of a system.

Let us first review the well-known relation between the impedance and the transient voltage which accompanies a pulse of current at time zero. For this purpose, we may make use of some of the relations given above. We shall assume that the current approaches as closely as we wish to a finite pulse of infinite height and infinitesimal length as discussed above. The corresponding $\Phi$ function then approaches a constant finite value. This means that the single-frequency components of the current are all cosines of the same infinitesimal amplitude. The corresponding components of the voltage are each made up of a cosine component proportional to the resistance and a sine component proportional to minus the reactance at the particular frequency. Formally these are the same as the components of the spectrum of a function of time. Whether or not they represent the spectrum of the transient voltage, depends on whether the impedance function is such that the integration involved in their summation has meaning. This will be true only if the impedance approaches zero at infinite frequency at a sufficiently rapid rate. In what follows we shall consider only systems which satisfy this condition.

We wish now to represent the spectrum of the transient in terms of the new analysis. Since the amplitudes of the voltage waves are $R(\nu)$ and $-X(\nu)$, $(\nu>0)$, those of their components of frequencies $\nu$ and $-\nu$ will be each half those values, so

$$\Phi(\nu) = \tfrac{1}{2}[R(\nu) - X(\nu)],$$

$$\Phi(-\nu) = \tfrac{1}{2}[R(\nu) + X(\nu)],$$

$$= \tfrac{1}{2}[R(-\nu) - X(-\nu)].$$

Also

$$\Phi_e(\nu) = \tfrac{1}{2}R(\nu), \tag{21}$$

$$\Phi_0(\nu) = -\tfrac{1}{2}X(\nu). \tag{22}$$

The $\Phi$ function corresponding to the transient voltage is then given by one half of the resistance minus the reactance. The application of this relation will next be illustrated by some practical examples.

*Echoes as Single-Instant Components*

To arrive at a more concrete picture of a single pulse, at a time other than zero, as corresponding to a sinusoidal function of frequency, let us consider what happens when a finite pulse of current of the kind just discussed is sent into a distortionless line at time zero by a generator of infinite internal impedance. First suppose the line to be of infinite length, or terminated in its own impedance, so that there is no reflected wave. The voltage across the input is also a pulse, the spectrum of which is made up of cosine components of uniform amplitude. This is consistent with the impedance of the line being a uniform resistance. Suppose now that the distant end be opened. The initial voltage pulse will then be followed by an echo, delayed by a period $t_1$ equal to twice the transmission time of the line. Owing to the infinite generator impedance no current will accompany this pulse and it will be reflected as from an open circuit. Let us assume that the line attenuation is great enough that subsequent echoes are negligible. The first echo then constitutes a finite voltage pulse at time $t_1$. The $\Phi$ function corresponding to it will be a finite sinusoid, for which the number of cycles per unit frequency range is given by $t_1$. This may be added to the uniform value of $\Phi(\nu)$ representing the initial pulse. Since the attenuation is assumed large, the sinusoidal part will appear as a ripple on the larger uniform value. When this resultant $\Phi$ function is resolved into its even and odd components, we have the functions which represent the resistance and minus the reactance of the open-circuited line. These will each have a sinusoidal component, the phases of the two being in quadrature. This picture of the line impedance will be recognized as that in common use for locating points of reflection by means of the sinusoidal variations in impedance.

Suppose now that we greatly reduce the attenuation. The initial pulse will then be followed by a long series of equally spaced pulses of gradually diminishing magnitude. Together they constitute the analog of a Fourier series, with $t_1$ as a fundamental. Each pulse will contribute to $\Phi(\nu)$ a sinusoidal component for which the number of cycles per unit frequency range is $nt_1$ and the wavelength on the frequency scale is $1/nt_1$. These combine to form a $\Phi$ function which has a sharp peak at the fundamental resonant frequency $1/t_1$ of the line. Also this $\Phi$ function repeats itself on the frequency scale in successive intervals of $1/t_1$ thus providing peaks corresponding to resonance at the harmonic frequencies. This point of view resembles very closely that used by Mason[3] in treating the steady-state properties of circuits in terms of multiple echoes.

[2] R. V. L. Hartley, "The transmission of information," *Bell Sys. Tech. Jour.*, vol. 7, pp. 535–563; July, 1928.

[3] W. P. Mason, "A new method for obtaining transient solutions of electrical networks." *Bell Sys. Tech. Jour.*, vol. 8; pp. 109–134; January, 1929.

*Distortion in Terms of Paired Echoes*

The correspondence between echoes and sinusoidal variations in impedance was carried further by MacColl[4] in a study of phase distortion. He noted that in certain circuits the departure of the phase characteristic from linearity was approximately sinusoidal. He therefore assumed small sinusoidal variations in the magnitude and phase of the transfer admittance and showed that the resulting transient distortion of the current could be represented by reduced replicas of the applied signal displaced in both directions along the time axis. More recently Wheeler[5] and Strecker[6] have represented the distortion of a television signal in terms of two groups of similar "paired echoes." In one group each pair corresponds to a sinusoidal component of the attenuation-frequency function of the system and in the other to a component of the phase-frequency function. The relations underlying this correspondence are approximate and the method is applicable only when the distortion is relatively small. Burrows[7] has suggested a method of successive approximations based on MacColl's results, which increases the accuracy and evaluates the residual error. He points out that in the amplitude case, the relations become exact if the amplitude itself is used, rather than its logarithm, the transmission loss, as used by Wheeler. Still more recently Strecker[8] has pointed out what is obvious from the present point of view, that the method of paired echoes may be made to give exact results for all values of distortion, if logarithmic relations are avoided and the system is described in terms of its transfer impedance, or admittance. As an alternative we may represent the system by its $\Phi$ function, and analyze this into its components of negative and positive instants, in accordance with (1). The amplitudes of these give directly the magnitudes of the resultant echoes which precede and follow the signal by particular intervals.

Another example of the representation of pulses on the time scale by sinusoids on the frequency scale will be found in Kallmann's[9] treatment of "transversal filters." By adjusting the amplitudes and signs of components corresponding to pulses having different arrival times, he constructs a function of frequency to fit the desired filter characteristic.

[4] L. A. McColl, "The distortion of signals by linear systems having amplitude and phase characteristics of a certain type," unpublished memoranda, December, 1931.

[5] Harold A. Wheeler, "The interpretation of amplitude and phase distortion in terms of paired echoes," Proc. I.R.E., vol. 27, pp. 359–384; June, 1939.

[6] F. Strecker, "Über den Einflus kleiner Phasenverzerrungen auf die Übertragung von Fernsehsignalen," *Elec. Nach. Tech.*, vol. 17, pp. 51–56; March, 1940.

[7] Charles R. Burrows and C. W. Carnahan, Discussion on Wheeler paper, footnote 5, Proc. I.R.E., vol. 27, pp. 384–386; June, 1939.

[8] F. Stecker, "Beeinflussung der Kurvenfarm von Vorgängen durch Dämpfungs- und Phasenverzerrung," *Elec. Nach. Tech.*, vol. 17, pp. 93–107; May, 1940.

[9] Heinz E. Kallmann, "Transversal filters," Proc. I.R.E., vol. 28, pp. 302–310; July, 1940.

*Representation of a Stable System by a Function of Positive Frequencies Only*

The pairs of functions of frequency which represent the steady-state characteristics of linear systems have been the subject of much study. Many years ago, MacColl questioned the need of two independent functions, or components of a complex function. He was able to show, in particular, that the susceptance of a passive circuit can be computed from its conductance. Work along this line has continued, and Bode[10] has extended these relations to include amplitude and phase. As a result of these studies it has come to be recognized that the performance of a passive system or a stable active one should be adequately described by a single function of positive frequencies. If so there should be, and there is, something in the nature of such systems which makes it unnecessary to specify the $\Phi$ function for both negative and positive frequencies.

This something is the fact that when a pulse is applied to such a system, the transient response is confined to the period following the pulse. In an unstable system the existence of a finite "response" is not dependent on a corresponding finite applied "stimulus." We saw above that the frequency characteristic may usually be interpreted as the spectrum of the transient response to an exciting pulse at time zero. If we call this response $f(t)$, $t>0$, then $f(-t)$ is always zero. From (9) and (10), then

$$f_e(t) = f_0(t) = \tfrac{1}{2}f(t),$$

and so either $f_e(t)$ or $f_0(t)$ alone is sufficient to determine $f(t)$. From (9), $f_e(t)$ is determined by $\Phi_e(\nu)$. But $\Phi_e(\nu)$ is an even function and so is completely described by its values for positive frequencies. Similarly from (10), $\Phi_0(\nu)$ is also adequate, and so either component of the familiar spectrum contains all the essential information. In more familiar language, the transient response may be deduced from either the real or the imaginary component of the steady-state response.

*Necessary Conditions for a Stable System*

The above result suggests some alternative formulations for the conditions which must be met by a steady-state characteristic if it is to correspond to a physically possible passive system or stable active system. When a pulse is applied to such a system at time zero, the transient $f(t)$ is zero for $t<0$. The amplitude of a single-instant component of the system characteristic, $\Phi(\nu)$, is $f(t)dt$. The condition, therefore, is that the amplitudes of all such components which correspond to negative values of $t$ shall be zero. Also, it follows from (13), that if $\Phi(\nu)$ be analyzed into sine components of variable amplitude and phase, in accordance with the familiar Fourier analysis, the condition requires that the phases of all the components be

[10] H. W. Bode, "Relations between attenuation and phase in feedback amplifier design," *Bell Sys. Tech. Jour.*, vol. 19, pp. 421–454; July, 1940.

$\pi/4$. Again, since the even and odd functions of time cancel for all negative instants they must be equal for all positive instants. Hence from (9), (10), (21), and (22), it must be true that

$$\int_{-\infty}^{\infty} d\omega \tfrac{1}{2}R(\omega) \cos \omega\, t_1 = \int_{-\infty}^{\infty} - d\omega \tfrac{1}{2}X(\omega) \sin \omega t$$

or

$$\int_{0}^{\infty} d\omega\, R(\omega) \cos \omega t = - \int_{0}^{\infty} d\omega\, X(\omega) \sin \omega t \tag{23}$$

for all positive values of $t$, if the characteristic is to be realizable. This relation is given by Guillemin.[11]

This criterion permits certain conclusions to be drawn regarding the paired echoes discussed by Wheeler and Strecker. In most transmission systems the reproduced signal is delayed by an interval $t_1$ to be determined by the slope of the linear component of the phase shift. Relative to this displaced zero of time the criterion says that all single-instant components of $\Phi(\nu)$ corresponding to instants before $-t_1$ must be zero. In order then for an assumed pair of variations in amplitude and phase to be realizable, it is necessary that, if the variation of one quantity corresponds to pre-echoes before $-t_1$, that of the other must correspond to equal echoes of opposite sign over this part of the time scale.

### *A Particular Necessary Condition*

While the satisfaction of (23) for all positive values of $t$ constitutes a necessary condition, its application to a particular case involves considerable labor. It is possible, by considering particular values of $t$, to deduce necessary conditions which are less general. As an example of this let us derive a relation which Bode[12] has established for the resistance of a circuit across the terminals of which there is a shunt capacitance $C$. The relation is

$$\int_{0}^{\infty} R(\omega)\, d\omega = \frac{\pi}{2C}\,.$$

We assume a pulse of current of infinitesimal duration to be sent into the circuit at time zero. The accompanying charge first accumulates in the shunt condenser and then proceeds to flow into the rest of the circuit. We select an instant $t_1$ so close to zero that the part of the charge which has left the condenser is negligible. The voltage at that instant is independent of the rest of the circuit and is determined solely by the condenser. So far as this instant is concerned, then, we may neglect the rest of the circuit. This is the equivalent of Bode's choice of a frequency so high that the impedance is equal to the capacitive reactance of the condenser. We then say that the voltage is to be zero at $-t_1$ and so (23) holds for that instant. By making $t_1$ small enough, $\cos t_1\omega$ may be made substantially equal to unity for any value of $\omega$. If we substitute $-1/\omega C$ for $X(\omega)$, we have

$$\int_{0}^{\infty} R(\omega)\, d\omega = \int_{0}^{\infty} \frac{\sin t_1\omega}{C\omega}\, d\omega,$$

$$= \frac{1}{C}\int_{0}^{\infty} \frac{\sin t_1\omega}{t_1\omega}\, d(t_1\omega) = \frac{\pi}{2C}\,.$$

A word of caution should perhaps be inserted regarding the application of this relation to physical circuits. The arrangement assumed neglects the inductance of the leads which must become appreciable as we approach infinite frequency. Once it does, the condenser is no longer shunted directly across the terminals. If this inductance is $L$, it can be shown by assuming the application of an impulsive voltage, that the conductance $G(\omega)$ is limited by the relation

$$\int_{0}^{\infty} G(\omega)\, d\omega = \frac{\pi}{2L},$$

instead of being infinite for all frequencies as was assumed. It is evident from this that if the impedance of an ideally lumped artificial line, having series $L$ and shunt $C$, is measured at mid-shunt, the integral of its resistance is limited, while at mid-series, that of its conductance is limited. However if we make it approach a uniform line by reducing $L$ and $C$, both of these limits approach infinity as they must if both $R$ and $G$ are to become constant for all frequencies.

*Note added in proof:* Since the above was written, a paper[13] has appeared in which Wheeler, in a treatment of unsymmetrical sidebands in terms of a zero-frequency carrier, makes use of functions of frequency which are defined independently for positive and negative frequencies, and of the resolution of these into even and odd components. The analysis, however, follows (7) and (8) above, and the object is not, as here, the development of a more symmetrical form of the Fourier identity.

[11] E. A. Guillemin, "Communication Networks," vol. 2, John Wiley and Sons, New York, N. Y., 1935, p. 503.

[12] H. W. Bode, United States Patent No. 2,242,878, 1941.

[13] H. A. Wheeler, "The solution of unsymmetrical-sideband problems with the aid of the zero-frequency carrier," Proc. I.R.E., vol. 29, pp. 446–458; August, 1941.