M. A. (KEN) CLEMENTS

TERENCE TAO

ABSTRACT. The article is a biographical account of Terence Tao's mathematical development. Born in 1975 he has exhibited a formidable mathematical precociousness which the author describes in some detail. The paper also presents the social and family context surrounding this precociousness and discusses the educational implications of this data.

1. INTRODUCTION

I first heard of Terence Tao on 27 April 1983, when an article on him appeared on the front page of the Adelaide daily morning newspaper, the *Advertiser*. The article was headed:

TINY TERENCE, 7, IS HIGH SCHOOL WHIZ

The article explained that Terence spent two-fifths of his school time at Blackwood High School, where he studied Year 11 Mathematics and Physics. He spent the remainder of his school time at Bellevue Heights Primary School. According to the article Terence learnt to read and write at the age of two by watching Sesame Street, and his teachers thought that while he had the academic ability of a 16-year-old, his maturity was that of a seven-year-old. Terence's mathematics teacher at Blackwood High School was quoted as saying that Terence fitted very well into the class and found the work easy. 'There is very little I actually teach him', the teacher said, 'he finishes all the work two lessons before the rest'. His primary school principal described him as 'a happy little fellow who has a clear understanding of the fact that he is different'. Terence's hobbies were said to include computing, playing with his electronics kit and reading science fiction novels such as *The Restaurant at the End of the Universe*. His father, Dr Billy Tao, a medical practitioner, was born in China and his mother, Mrs Grace Tao, a graduate in Physics and Mathematics, was born in Hong Kong. The parents met at the University of Hong Kong, where both were educated before emigrating to Australia in 1972. They have two children younger than Terence, Trevor and Nigel.

Having been interested in exceptionally capable children in mathematics for many years (during my eight years at Monash University, 1974–1982, I gave many lectures on the subject; also, I both carried out research of my own and supervised higher degree research in the general area), I read the *Advertiser* article with interest. 'At least', I thought 'the parents and the teachers involved

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were courageous enough to attempt something designed at meeting Terence's special needs'. However, since leaving Monash University in February 1982, to begin a Bible college course in Adelaide I had resolved not to become involved in mathematics education matters anymore, I resisted the temptation to contact the Tao family.

In June 1983 I was invited to address an in-service education conference for teachers on 'the identification of exceptionally gifted children in mathematics'. I agreed to do so (somewhat reluctantly, because of the afore-mentioned resolve). Soon after the beginning of my talk at the conference I made a passing reference to the *Advertiser* article on Terence. When I had finished speaking one of the conference participants introduced himself to me as Terence's father, Billy. Dr Tao invited me to his home to speak to Terence, and to carry out an assessment of his mathematical abilities and performance. How could I refuse?

2. THE INITIAL ASSESSMENT

I went to Terence's home on the 16 July 1983, the day before his eighth birthday. When I arrived Billy introduced me to his wife, Grace, and then to Terence, who had been sitting in the far corner of a room reading a hardback book with the title *Calculus*. Terence was small, even for a seven-year-old. After meeting his two brothers, I was accompanied by Terence to his father's study, where, after a brief chat, I began my usual assessment procedure for exceptionally bright primary school-age children. I asked Terence to attempt the 60 questions on Australian Council for Educational Research's *Operations Test* (Cornish and Wines, 1977).

Before Terence began the *Operations Test* I told him that he'd find most of the early questions easy, but said 'you shouldn't laugh at the questions, because they get harder towards the end of the test'. I was intrigued by his reply: 'the questions won't know if I laugh at them, because they haven't got ears'.

Terence got 60/60 on the *Operations Test*. As I watched him solve the problems it became apparent to me that the test was far too easy for him. The following shows his working for Question 58 of the test:

Question 58. If $(p \div q) \div r = \Delta \div (q \div r)$, find Δ.

Terence wrote: $\dfrac{p/q}{r} = \dfrac{\Delta}{q/r}$

$$\frac{p}{qr} = \frac{\Delta}{q/r}$$

$$\frac{p}{r} = \frac{\Delta}{1/r}$$

$$\frac{p}{r} = \Delta r$$

$$\frac{p}{r^2} = \Delta.$$

According to ACER norms for the *Operations Test*, an average Year 12 student could be expected to get a score of 53/60 on the test (see Cornish and Wines, 1977, pp. 21 and 38). Although I had given the test to many very bright primary-school-age children before, none of them had ever got more than 57/60 – and Terence was probably the youngest person I had ever asked to do the test.

QUESTIONS (to be presented to the students in written form; answers should be worked out *mentally*)

S1 Two circles have radii equal to 2 cm and 3 cm. The distance between their centres is 4 cm. Do they intersect?

S2 What angle does an hour hand describe in 20 minutes?

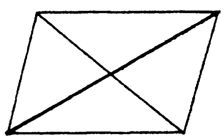
A1 A can of kerosene weighs 8 kg. Half the kerosene is poured out of it, after which the can weighs $4\frac{1}{2}$ kg. What is the weight of the empty can?

V1 What time is it now if the time which passed since noon constitutes a third of the time that remains until midnight?

M2 I walk from home to school in 30 minutes and my brother takes 40 minutes. My brother left 5 minutes before I did. In how many minutes will I overtake him?

A The perimeter of a right angled triangle is 5 cm. Two of its sides are each 2 cm long. How long is the third side?

D How many triangles are there?



E. A class received some regular and some special notebooks, and altogether there were 80 notebooks. A regular notebook costs 20 cents and a special one 10 cents. How many of each kind of notebook did the class receive?

Fig. 1. Eight questions, given in writing but to be solved mentally.

Suitably impressed, I then showed Terence the set of written questions in Figure 1, and asked him to solve them for me, 'in his head', without writing anything down. Terence was instructed to speak out his thoughts, and as he did so I recorded, in writing what he said. The questions are all from Krutetskii (1976), and the symbols at the beginning of the questions correspond to Krutetskii's classifications of them. Here is what Terence said when answering the questions.

Question 1: Yes. If they didn't intersect the distance between their centres would be more than 5. (Terence then used hand movements to explain his answer.)

Question 2: Simple. 1/3 of 1/12th of a full circle is 1/36th of a circle. 1/36th of 360° equals 10°.

Question 3: You get an algebraic equation, but it's hard to work out in your head.

Weight of Can + Weight of Kero = 8
Weight of Can + $\frac{1}{2}$ (weight of Kero) = $4\frac{1}{2}$
So Weight of Kero = 7 kg wt
Weight of Can = 1 kg wt.

Question 4: 1 unit + 3 units = 12 hours
So 1 unit = 3 hours
So time is 3 p.m.

Question 5: 35 minutes. If you started at the same time as your brother you'd arrive 10 minutes before him . . . Oh no. 15 minutes, because then you'd both be halfway.

Question 6: The third side is 1 cm . . . That can't be true, by the way. Pythagoras' Theorem says it has to be . . . $\sqrt{8}$ or . . . it's impossible.

Question 7: 8 triangles.

Question 8: I don't know really (laughs).

$$r + s = 80.$$

All you're given is the cost.
It can't be done.
Could be 40 regulars and 40 specials, or 50 regulars and 30 specials.

Terence answered all the questions, verbally, in a total time of 9 minutes.

He was the first primary-school-age child I had tested to get all eight questions 'correct'.

When Terence had been answering the questions on the A.C.E.R. *Operations Test* I had noticed that he often justified an algebraic step by writing the appropriate algebraic law (e.g., associative law for $\times n$). This prompted me to vary my normal testing procedure. After Terence had completed the eight 'Krutetskii' problems the following conversation took place (M.A.C. = author; T.T. = Terence):

M.A.C.: What is the associative law for addition of real numbers?
T.T.: It doesn't matter where you put brackets: $a + b \ldots + c$ equals $a + \ldots b + c$.
M.A.C.: What about the commutative laws?
T.T.: You can juggle the order: $a \times b = b \times a$
$a + b = b + a$
M.A.C.: What is a group?
T.T.: A set which is mapped onto itself by a binary operation. The binary operation is associative, and the set has an identity e such that $e \times x$ equals x for all x in the set. Also, for each x in the set there is an inverse x' in the set such that $x' * x$ equals e.
M.A.C.: What about the commutative law?
T.T.: Holds for Abelian groups.
M.A.C.: What is a field?
T.T.: I don't know.
M.A.C.: What is the distributive law?
T.T.: $*$ distributes over $\circ$;
$a * (b \circ c)$ equals $(a * b) \circ (a * c)$
M.A.C.: Give me an example.
T.T.: Multiplication over addition.
M.A.C.: Addition over multiplication?
T.T.: Only for Boolean algebras.

I was quite impressed by all this. Not only did he have an astounding grasp of algebraic definitions, for someone who was still seven years old, but I was amazed at how he used sophisticated mathematical language freely.

The next question I asked was also from Krutetskii. The question, together with Terence's written solution to it, is shown in Figure 2.

I was beginning to form the impression that Terence preferred to use analytic, non-visual methods in preference to making extensive use of visual imagery (see Lean and Clements, 1981, pp. 280–288).

The length of each side of a square is increased by 3 m. The area of the 'new' square is 39 m^2 more than that of the original square. How long are the sides of the 'new' square?

Terence's solution

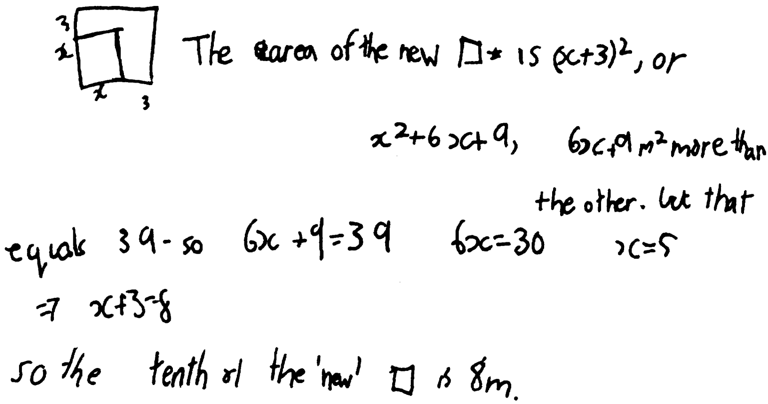


Fig. 2. Terence's solution to a Krutetskii problem (16 July 1983).

We decided to break for afternoon tea, which occupied about 45 minutes. Terence was then happy to return with me to his father's study for further questioning. Once back in the study I gave him the three questions shown in Figure 3 and told him to write his solutions, in full, on paper.

These are the 'solutions' Terence wrote for the three questions:

1 Suppose you decided to write down all whole numbers from 1 to 99,999. How many times would have have to write the number 1?

2 A car travelled from A to B at 20 km/hr and back at 30 km/hr. What is the car's average speed for the whole trip?

3 In a supermarket there are 24 sacks of potatoes left, some of which weigh 9 kg, and the others 15 kg. The potatoes in the 9 kg sacks are smaller than those in the 15 kg sacks, and each of the 24 sacks contain exactly the same number of potatoes.

If the total weight of all the 15 kg sacks equals the total weight of all the 9 kg sacks, how many 9 kg sacks are there?

Fig. 3. Written questions, requiring written answers.

Question 1. (Terence gave the following incorrect solution)

1 digit total 1
2 digit total $9 + 1 = 10$
3 digit total $99 + 9 + 1 = 109$
4 digit total $999 + 99 + 9 + 1 = 1018$
5 digit total $9999 + 999 + 99 + 9 + 1 = 11\,017$

There will be 12 334 1's from 1 to 99 999.

Question 2. Let $D(\overline{AB}) = x$.

Then

$$\text{av. speed} = \frac{\text{total distance}}{\text{total time}}$$

$$\text{distance} = 2x$$

$$\begin{aligned}\text{time} &= \frac{x}{20} + \frac{x}{30} \\ &= \frac{50x}{600} \\ &= \frac{5}{60}x\end{aligned}$$

$$\begin{aligned}\text{average speed} &= \frac{2x}{\frac{5}{60}x} \\ &= \frac{2}{1/12} \\ &= 24\text{ km/h, assuming } x \neq 0\text{, i.e., } A \text{ and } B \text{ are not in the same position.}\end{aligned}$$

Question 3.

$$\begin{aligned}x + y &= 24 \\ 9x &= 15y \\ 3x &= 5y \\ x &= \tfrac{5}{3}y \\ \tfrac{8}{3}y &= 24 \\ \tfrac{1}{3}y &= 3, \quad y = 9, \quad x = 15\end{aligned}$$

So there are 15 9 kg sacks.

Terence's solutions to Questions 2 and 3 strengthened my conviction that he preferred to make use of analytic, non-visual solution strategies. While his attempted solution to Question 1 contained arithmetic errors, the strategy he applied was sound enough, though there are more elegant methods which could be used: e.g., the number of ones would equal $(100\,000 \times 5) \div 10 = 50\,000$.

After Terence had finished writing his correct solution to Question 2, he looked puzzled, and the following conversation ensued:

T.T.: You could say the average of 20 and 30 is 25?

M.A.C.: Which is right 25 or 24?

T.T.: 25?

M.A.C.: So, what's wrong with your working? Have you made a mistake when you got 24 km/h?

T.T.: Yes.

Perhaps my mode of questioning pushed him to say that 25 km/h was the correct solution.

When Terence had completed his solution for Question 3 I asked him what he thought of the question. He told me 'there's one piece of information you don't need – where it says "the potatoes in the 9 kg sacks are smaller" '.

By this stage Terence was showing slight signs of fatigue (though his interest was still high), so I decided to ask him only two more, relatively simple questions. First, I asked him to sketch the graph of $y = x^2 + x$, which he did, immediately. I asked him to find the co-ordinates of the turning point, and he wrote

$$\frac{dy}{dx} = 2x + 1$$

$$x = -\tfrac{1}{2}, \quad y = -\tfrac{1}{4}$$

$$(-\tfrac{1}{2}, -\tfrac{1}{4}).$$

This response took about 20 seconds.

I then asked him to sketch $y = x^3 - 2x^2 + x$. His rather untidy response is shown in Figure 4.

Terence's response took about one minute. It is interesting to observe that he had not yet begun to study calculus at school.

Additional questioning revealed that Terence had a sound grasp of most topics in traditional school mathematics up to and including that expected of Year 11 students. He also understood, and could apply, the first principles and rules of differential calculus.

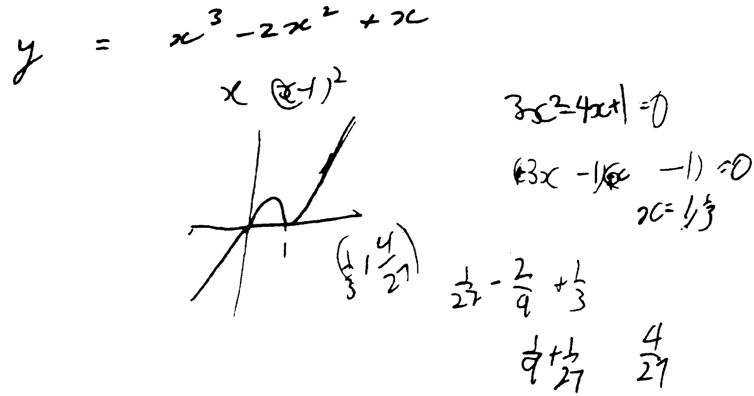


Fig. 4. Terence's response to request to sketch the graph of $7 = x^3 - 2x^2 + x$ (16th July 1983).

Before leaving the Tao household I spoke to Dr and Mrs Tao about their backgrounds and their attitudes towards Terence and his intellectual development. Mrs Tao (Grace) has taught Science, Physics, Chemistry and Mathematics in secondary schools in Hong Kong and Australia. She said that while she sometimes attempts to guide Terence's mathematical learning, she doesn't help him much because 'he doesn't like to be told what to do in mathematics'. She recalled that one night, in 1983, when Terence was thinking about how to evaluate the continued fraction

$$1 + \cfrac{2}{1 + \cfrac{2}{1 + \cfrac{2}{1 + \dots}}}$$

She had said to him: 'try a quadratic'. Immediately Terence had written

$$x = 1 + \frac{2}{x}$$

$$x^2 - x - 2 = 0$$

$$x = 2 \quad \text{or} \quad -1$$

$$x = 2 \quad (\text{must be } + ve).$$

Mrs Tao's role, then, is more one of guiding and stimulating Terence's development than one of teaching him. She said that Terence likes to read mathematics by himself, and he often spent three or four hours after school reading mathematics textbooks.

I made arrangements to come back in order to continue my assessment of Terence. As I was leaving Billy showed me some of Terence's efforts, over the last two years, on the family's Commodore computer. Terence had taught himself BASIC language (by reading a book) and had written many programs on mathematics problems. Some of the names of his programs were 'Euclid's algorithm', 'Fibonacci' and 'Prime Numbers'. His 'Fibonacci' program, shown in Figure 5, is interesting in that a careful reading of it will reveal something of Terence's creative, lively personality. Also, it is fascinating to observe that Terence wrote many of his programs early in 1982, when he was 6 years old.

```
8 print "J"
10 print "here comes mr. fibonacci"
20 print "can you guess which year was mr. fibonacci born?"
30 print "write down a number please . . . ": input c
31 if c = 1170 then print "you are correct! now we start": go to 150
50 if c > 1250 then print "no, he is already in heaven, try again": go to 30
60 if c < 1170 then print "sorry, he wasn't born yet! try again": go to 30
70 if c > 1170 < 1250 then print "he would be ";c—1170;" years old"
71 print "now can you guess?": input c
72 if c = 1170 then 31
73 print "you are wrong. try again.": go to 71
150 print "up to which number do you want me tell you all the fibonacci numbers"
151 input n
160 print "J"
190 print "okay. here they go!"
200 s = 1
210 t = 1
220 if s > = n then 270
230 if t > = n then 270
240 print s; t;
250 s = s + t
260 t = t + s
265 go to 220
270 print
271 print "another game, while fibonacci is waiting (y), or no more? (n)": print
272 get c$: if c$ = " " then 272
273 if c$ = "y" then 150
274 if c$ = "n" then 300
280 go to 272
300 print "mr. fibonacci is leaving now,"
310 print "and wishes to see you again sometime in the future"
312 print
313 print
315 print "here goes his car!!!!!!!!"
320 print "(brmmmm-brmmmm-putt-putt-vraow-chatter-chatter bye mr. fibonacci!)"
390 go to 450
410 print
420 print
445 next i
450 end
```

Fig. 5. Terence's 'Fibonacci' program.

3. THE SECOND ASSESSMENT

Five weeks after I had first worked with Terence I returned to the Tao house (on 20th August 1983). He was now eight years of age, and during the five weeks I had learnt that he had gained 19th place out of about 2000 South Australian Year 11 entrants in a national school mathematics competition. He had sat for the competition examination in June 1983 (when he was seven). The fact that many schools encourage only their better students at mathematics to enter the competition added further merit to Terence's performance.

Once again, my assessment of Terence took place in his father's study. To begin, I asked Terence to consider whether

$$S = \{a + b\sqrt{2}: a, b \in R\}$$

is a group under the operation of 'addition'. He immediately showed that $(S, +)$ was a group. I then asked him if $(S, +, \times)$ was a field. His written reply was as follows:

> $(S, +)$ is an Abelian group (last question).
>
> *For* $\times$, Assoc, Commutative laws hold (properties of real numbers)
> $1 = 1 + 0\sqrt{2}$ is $\times$-identity
>
> $\times$-*inverse*
>
> $$\frac{1}{a + b\sqrt{2}} \cdot \frac{a - b\sqrt{2}}{a - b\sqrt{2}} = \frac{a}{a^2 - 2b^2} - \frac{b}{a^2 - 2b^2}\sqrt{2},$$
>
> so every el. in S has $\times$-inverse in S except 0.
>
> Distributive law holds (properties of real numbers).
> Thus $(S, +, \times)$ is a field.

I deliberately asked about a field, because it will be recalled, during the initial assessment Terence had told me he did not know what a field was. I was impressed that he had obviously taken the trouble to remedy this situation; further, the sophistication and succinct nature of his response on this occasion was something of which a university student in mathematics should have been proud.

Next I tested Terence's knowledge of some standard results and concepts in integral calculus. He could tell me antiderivatives of x^2, $\sqrt{x}$, $\sin x$, $\sec^2 x$, $1/(1 + x^2)$, $1/\sqrt{(1 - x^2)}$, but when asked for an antiderivative of $1/x$ he told me that he had 'not got up to that yet' in his reading. When I asked him to find an antiderivative of $1/(1 - x^2)$ he used the substution $x = \cos\theta$ to show that

$$\int \frac{dx}{1 - x^2} = \int -\operatorname{cosec}\theta \, d\theta.$$

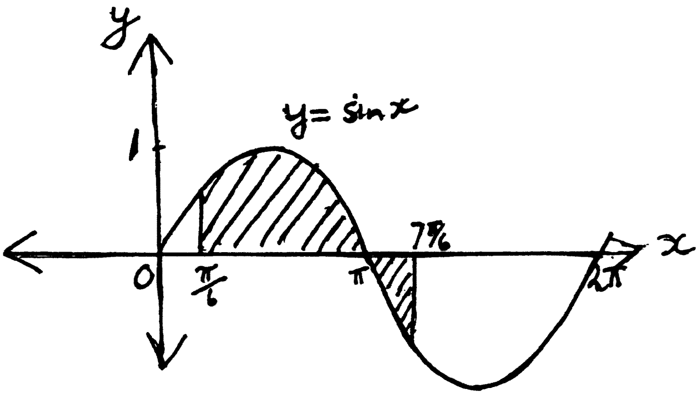


Fig. 6. An integration problem.

He then said that he couldn't do this. I mentioned the words 'partial fractions' to him, but this didn't help. He said he would read more on integration during the next few weeks.

I then drew, freehand, the sketch shown in Figure 6 and asked him to find the shaded area. He immediately wrote

$$\int_{\pi/6}^{\pi} \sin x \,\mathrm{d}x + \int_{\pi}^{7\pi/6} (-\sin x)\,\mathrm{d}x$$

$$= [-\cos x]_{\pi/6}^{\pi} + [\cos x]_{\pi}^{7\pi/6}$$

$$= 1 + \sqrt{3}/2 - \sqrt{3}/2 + 1$$

$$= 2.$$

When asked to find the area between the graph of $y = 1/x^2$ and the x-axis, for all $x \geqslant 1$, Terence wrote

$$\int_{1}^{\infty} \frac{\mathrm{d}x}{x^2} = [-1/x]_1^{\infty} = 0 - (-1) = 1.$$

Terence then attempted the Monash *Space Visualization Test* (see Wattanawaha and Clements (1982) for examples of items on this test), and obtained a score of 27/30. Norms for the test indicate that the mean score of Year 12 students is 24/30. One of the three questions which Terence got wrong is shown in Figure 7.

After Terence had completed the test I asked him to verbalize the methods he had used when attempting the questions on the test, and these verbal responses were taped. With respect to the question in Figure 7 he explained

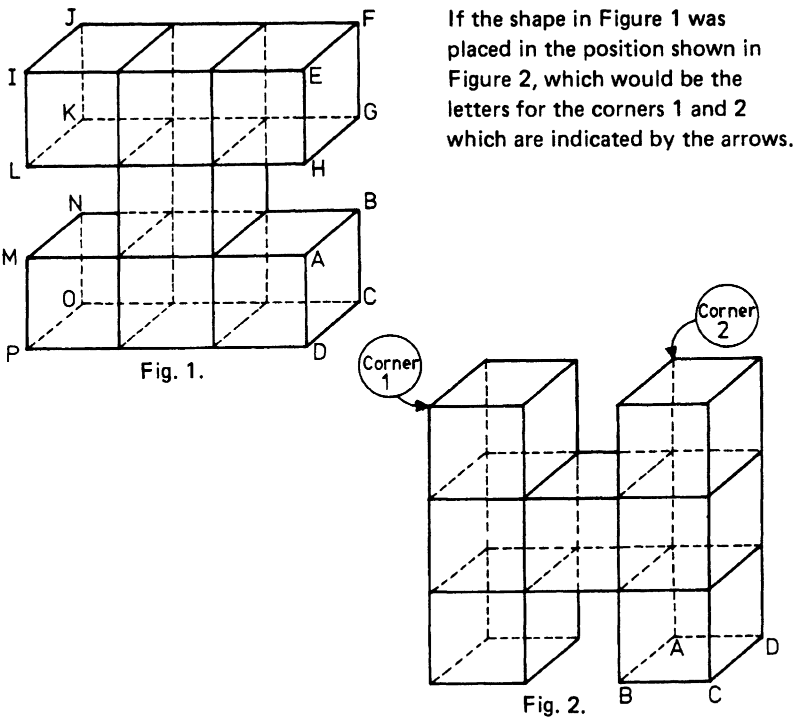


Fig. 7. A question from the Monash *Space Visualization Test*.

that he had used two mental rotations to try to move Figure 1 into exactly the same position as Figure 2. He correctly identified Corner 1 as J, but thought that Corner 2 was N. He told me he found it difficult to carry out the required visualization exercise. One of his other errors on the test was also due to his inability to carry out reasonably complex manipulations of visual images.

Analysis of the methods Terence said he used when attempting the questions on the *Space Visualization Test* strongly suggested that he preferred to use non-visual, analytic methods whenever these occurred to him, even if they required more complicated thinking than more visual methods which could be used. Thus, for example, for the question shown in Figure 8, he said that he checked each shape by the reflection law (each point has an image on the other side of the mediator), and he did *not* imagine each shape being folded along the dotted line.

Terence's performance on the *Space Visualization Test* suggested that his spatial abilities are exceptionally well developed. However, on questions which can be done by more analytic, less visual methods, he is happy to use these in

QUESTION 9. Which of A, B, C, D, E *cannot* be folded along the dotted line so that one half fits exactly over the other half?

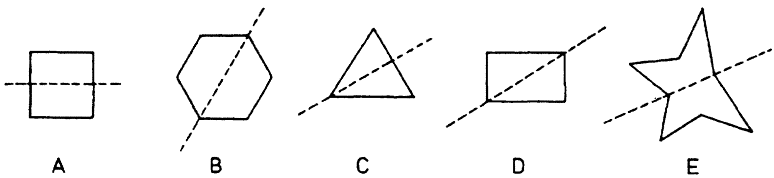


Fig. 8. A question which Terence did by an analytic method.

preference to methods which require manipulation of visual imagery. Burden and Coulson (1981), after a detailed study of methods used by persons attempting a variety of spatial tasks (from widely used spatial tests), have reported that persons who prefer analytic to more visual methods tend to perform better on spatial tests. Thus, while Terence does experience some difficulty when attempting complex manipulations of visual images, his preference for more analytic methods served him well on the *Space Visualization Test*. In this context, it is interesting that Krutetskii (1976, p. 351) claimed that neither an ability for spatial concepts nor an ability to visualize abstract mathematical relationships are obligatory in the structure of mathematical giftedness. However, the degree of their development in an individual does influence that individual's mathematical cast of mind (see also Shepard, 1978, pp. 133–184).

While Terence was attempting the *Space Visualization Test* I made up a list of 22 books on mathematics which, according to records he kept, he had ready over the past two years. Among these books were:

K. K. Ko — *Matrices and Vectors*. Hong Kong: 1971
Numbers, Inequalities, Linear Programming. Hong Kong: 1971
E. A. Abbott — *Flatland*. New York (Dover): 1952
Irving Adler — *Readings in Mathematics I, II*. Lexington (Mass.): 1972
S. F. Barker — *The Elements of Logic*. New York: 1974
S. L. Greitzer — *International Mathematical Olympiads 1959–1977*. Washington D.C.: 1978
A. J. Sherlock, E. M. Roebuck, M. G. Godfrey — *Calculus: Pure and Applied*.

Terence tends to read whole books rather than parts of books. He is keen to receive advice on which books he should read next. His father told me that he has a remarkable memory for virtually everything he reads. On several occasions when I spoke with Terence about mathematics he punctuated the conversation by saying 'Oh yes, I've read about that'. He then went and got a book, quickly found the relevant section, and showed it to me.

After Terence had completed the spatial test I then gave him an open-ended task involving the following sequences, in which each term after the first is the sum of the squares of the digits in the preceding term. Figure 9 shows the information Terence was given and the questions which he was asked to answer.

1 → 1 ↻
2 → 4 → 16 → 37 → 58 → 89 → 145 → 42 → 20 → 4
3 → 9 →
4
5
6
7 → 49 → 97 → 130 → 10 → 1 ↻
8

Questions:

1 Which natural numbers produce sequences 'like' those for 2 and 3?
2 Which natural numbers produce sequences 'like' those for 1 and 7?
3 Which natural numbers produce sequences which are not 'like' those for 1 or 7?
4 Any other points of interest.

Fig. 9. An open-ended task.

Terence was allowed about twenty minutes on this task. He quickly established that 4, 5, 6, 8 and 9 produced sequences 'like' 2 and 3. He stated that no natural number would produce a sequence 'different from' the two which were already obvious, but did not offer a proof of this conjecture. He did not show any evidence of having considered the kinds of sequences produced by natural numbers with two or more digits. For Question 4 he raised the interesting question of whether similar patterns would hold for arithmetics other than base 10 arithmetic. This constituted Terence's total reply and, I must confess, I was disappointed that he did not provide a longer, more profound analysis of the situation.

For the second assessment I had been accompanied to the Tao household by Dr Max Stephens, Principal Curriculum Officer in the Curriculum Branch of the Education Department of Victoria. I asked Dr Stephens if he would like to ask Terence a question. Dr Stephens has provided the following report on what ensued:

I drew pictures of the 6 Australian coins: 1 cent, 2 cent, 5 cent, 10 cent, 20 cent and 50 cent, and then asked Terry how many different totals he could make using the coins. He replied 720, but then added, "They will all be the same." I realised that my question should have indicated that the coins could be taken one at a time, two at a time, three at a time, up to all six at once. Having heard this question rephrased, Terry said, "There are $2^6 - 1$ ways of making totals out of these six coins". I asked him whether he was familiar with the notation for writing a selection of one or two or more things from a group of six. He said that he was. We then wrote down the six possible groupings of the coins involved, and showed that the result was 63. He had already obtained that result on his own, using the formula $2^6 - 1$. I said to him, "Perhaps, some of these groupings give the same total as other groupings. What do you think of that possibility?" Straight away, he replied, "That can't be so. If you take any coin, its value is greater than the total of all the coins smaller than it".

We then had afternoon tea. Terence seemed happy to continue working with me, so after afternoon tea I asked him to attempt the following well-known addition problem.

$$\begin{array}{r} \text{A} \\ \text{M E R R Y} \\ \text{X M A S} \\ \hline \text{T U R K E Y} \\ \hline \end{array}$$

He was told that the letters represented the digits 0, 1, 2, . . . 9, that $K = 3$, and that whenever a letter appears more than once it must take the same value for each appearance. The problem is to find the value for each letter. I asked Terence to verbalise his thoughts as he attempted the problem, and his verbalizations were taped for later analysis.

Terence quickly solved the problem correctly. The most interesting feature of his strategy was an obvious liking for writing down and solving relevant simultaneous equations. Once again, his preference for using an analytic, highly logical problem-solving strategy was revealed.

This completed my second assessment session with Terence. After the session Mrs Tao provided the summary of Terence's school timetable for Term 3, 1983 (see Table I). The entries marked with an asterisk (*) were to take place at Bellevue Heights Primary School (Year 5) and the others at Blackwood High School (year 8: General Studies, Year 11: Physics, Year 12: Mathematics). Mrs Tao would provide the necessary transport between schools.

Because Terence had already studied all of the topics which would have been taught in Year 11 Mathematics at Blackwood High School in term 3, it had been decided that he should attend Year 12 Mathematics classes, at the School, during the term.

At my request Dr Tao provided me with copies of three reports, by a

TABLE I

Terence's School Timetable for Term 3, 1983

Time	Monday	Tuesday	Wednesday	Thursday	Friday
9:00– 9:45	Maths 2	Maths 2	Spelling*	Maths 2	Maths 1
9:45–10:30	Maths 2	Physics	Reading*	Maths 1	Library
10:30–11:15	Maths 1	Library	Physics	Physics	Physics
RECESS					
11:30–12:15	Gen. Stud.	Maths 1	Library	Gen. Stud.	Drama*
12:15– 1:00	Fitness*	Fitness*	Maths 1	Fitness*	Fitness*
LUNCH					
1:40– 2:25	Physics	Soc. Stud.* Music*	Maths 1	Health Sc.*	Art*
2:25– 3.10	Soc. Stud.*	Phys. Ed.*	Maths 2	Poetry*	Maths 2

clinical psychologist, on Terence. These reports were based on data gained at interviews with Terence when he was 4 years 7 months (February 1980), 5 years 9 months (May 1981), and 6 years 4 months (November 1981).

In the first report the clinical psychologist stated that although Terence was only $4\frac{1}{2}$ years old he was functioning intellectually more like an 8 to 10 year-old. He added that Terence would require careful supervision during his schooling to see that his intellectual, social and emotional needs were met adequately. In the second report the psychologist stated that Terence was in the 95th percentile range for 11 year olds on the Raven's Controlled Projection Matrices test (a primarily non-verbal test of reasoning). In the third report Terence, at age 6 years 4 months, is said to have gained maximum or near maximum scores on the Wechsler Intelligence Scale for children, with there being no difference between his verbal and performance (practical, non-verbal) intelligence. His overall Mental Age was 14 years (very superior range of intellect for a 6 year-old). The psychologist indicated that while the situation seemed quite favourable at that time, with Terence accepting normal progression through the school grades, special arrangements might have to be made for his transition to secondary and tertiary education.

I first met Terence almost twenty months after the third report was written. Much had been done during this period and special arrangements had been, and were being, made for his secondary and tertiary education. I had to admire the efforts which his parents, Billy and Grace, had made on his behalf, despite the danger that they would be labelled 'pushy' by persons who did not understand. As Julian Stanley and Camilla Benbow (1982, p. 8) have noted, there is great hostility towards precocious intellectual achievement in many quarters.

The following statement is a challenge, not only to educators but to the whole community:

> Why is a child violinist, composer, chess player, cinema star or athlete lauded, whereas the child who excels mathematically or writes splendid poetry is sometimes regarded as a "freak"? This attitude may be stronger in the United States than in some other countries such as the Soviet Union and China. Whether or not it is, however, the deleterious influence on intellectual achievements is probably great. Furthermore, many people consider attempts to provide special educational opportunities for the intellectually talented as elitist. This, we believe, is based on a misconception: democracy does not mean that children must receive the same education, but instead that they should have equal opportunities to develop their abilities.
>
> (Stanley and Benbow, 1982, p. 8)

In a society where hostility towards parents who regard their children as sufficiently bright to warrant extra-special educational consideration is endemic, it is refreshing to discover parents as courageous and realistic as Grace and Billy Tao.

4. THE THIRD ASSESSMENT

The Tao's invited me to their home on 17 September 1983 in order to join them in discussions with Dr Tom van Dulken, a senior tutor in the school of mathematics sciences of Flinders University (Adelaide), concerning the possibility of Terence's early entry to Flinders University.

After Dr van Dulken had spoken with Terence, mostly on aspects of various mathematical topics, I asked Terence a few more questions. Terence found, at my request, antiderivatives of $x \sin x$ and $e^x \cos x$. I asked him to find an antiderivative of $\sin x/(\sin x + \cos x)$ and was impressed by his written reply:

$$\int \frac{\sin x}{\sin x + \cos x}\,dx = \int \left[\frac{1}{2} - \frac{\cos x - \sin x}{2(\sin x + \cos x)}\right] dx$$
$$= \tfrac{1}{2}x - \tfrac{1}{2}\ln|\sin x + \cos x| + C.$$

I noticed that he now knew that $\ln|x|$ is an antiderivative of $1/x$ (something he had not known at the time of the second assessment).

When I asked Terence to find the constant form in the binomial expansion of $(2x - 4/x)^{10}$ he told me that he had not yet done much on the binomial theorem and proceeded, laboriously, to construct Pascal's triangle in order to obtain an answer. I told him not to worry about it, but to find out how to do it quickly before I saw him 'next time'. A couple of weeks later, when the Tao family visited my home, I asked Terence to find the constant term in the binomial expansion of $(2x - 5/x)^{10}$. He told me he *could* do it quickly now, and wrote:

$$(n+1)\text{th term} = \binom{10}{n}(2x)^n(-5/x)^{10-n}$$

$$= \binom{10}{n}2^n(-5)^{10-n}x^{2n-10}.$$

When $n = 5$ the constant term is

$$\binom{10}{5}2^5(-5)^5 = 252 \times (-10)^5$$

$$= -25\,200\,000.$$

Indeed, Terence could now do such problems quickly.

Since Terence had already been speaking with Dr van Dulken for some time before I started with my questions, I decided not to ask him any more questions. However, he agreed to my borrowing an exercise book which contained some of the mathematics exercises he had done at home over recent weeks. On examining this book I found that often Terence wrote the date at the bottom of the page, and this enabled me to see that on many days he had done from three to five pages of work (by himself, at home). The following solutions, taken from the exercise book, indicate the level of work which Terence had been doing.

1. $$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 5y = 0, \quad y = 3, \quad \frac{dy}{dx} = -1 \quad \text{when } x = 0$$

$$K^2 - 6K + 5 = 0$$

$$(K-5)(K-1) = 0$$

$$K = 5, 1$$

$$y = A\,e^{5x} + B\,e^x \qquad\qquad A + B = 3$$

$$\frac{dy}{dx} = 5A\,e^{5x} + B\,e^x \qquad\qquad 5A + B = -1$$

$$A = -1, \; B = 4$$

$$y = 4e^x - e^{5x}$$

2. $$\int \frac{dx}{1 + \sin x + \cos x} \qquad t = \tan \tfrac{1}{2}x$$

$$= \int \frac{\frac{2}{1+t^2}\,\mathrm{d}t}{1 + \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}}$$

$$= \int \frac{2\,\mathrm{d}t}{1 + \cancel{t^2} + 2t + 1 - \cancel{t^2}}$$

$$= \ln|1+t| + C$$

$$= \ln|1 + \tan \tfrac{1}{2}x| + C$$

3.

$$\frac{3(x+1)}{x^2(x^2+3)} = \frac{Ax+B}{x^2} + \frac{Cx+D}{x^2+3}$$

$$3x + 3 = Ax^3 + Bx^2 + 3Ax + 3B + Cx^3 + Dx^2$$

$$3x + 3 = (A+C)x^3 + (B+D)x^2 + 3Ax + 3B$$

$$A = 1, \quad B = 1, \quad C = -1, \quad D = -1$$

$$\int \frac{3(x+1)}{x^2(x^2+3)}\,\mathrm{d}x = \int \frac{x+1}{x^2}\,\mathrm{d}x - \int \frac{x+1}{x^2+3}\,\mathrm{d}x$$

$$= \int \frac{\mathrm{d}x}{x} + \int \frac{\mathrm{d}x}{x^2} - \frac{1}{2}\int \frac{2x}{x^2+3}\,\mathrm{d}x - \int \frac{\mathrm{d}x}{x^2+3}$$

$$= \ln|x| - \frac{1}{x} - \frac{1}{2}\ln(x^2+3) - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C$$

His use of partial fractions in this last example is interesting when it is recalled that during my assessment of him, on 20 August 1983, he had not been able to find an antiderivative of $1/(1-x^2)$. Terence learns fast.

With respect to Terence's future schooling, Billy and Grace Tao have decided that in 1984 he will not study any mathematics at school, but will continue to work at home in such areas as algebraic structure, probability and statistics, computing, and analysis. In 1984 he will spend all his school time at Blackwood High School, where he will study humanities subjects in Year 8 classes, Geography in Years 10 and 11, Chemistry in Year 11 and Physics in Year 12. Provided Terence's interest in academic maths remains, and he appears to be socially and emotionally ready, he will begin a degree course in mathematics at Flinders University in 1985. Dr van Dulken believes that, even though he will be nine years old at the beginning of his university career (assuming everything goes as planned), he would be far more advanced mathematically than most, if not all, of his fellow first-year students; special