A Simple Randomization Procedure

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Summary

The paper describes a randomization procedure consisting in distributing a deck of cards into 10 decks using random decimal digits and repeating this step with each deck consisting of three or more cards. One random digit is used for randomizing a deck of two cards. This procedure, which is essentially a particular case of a general procedure described by Rao (1961), is called the multistage randomization procedure, or MRP. Some applications are described. A recursive formula is given for the expected number of random digits required by MRP for the randomization of n symbols. A measure of the efficiency of a randomization procedure is presented. The efficiency of MRP is compared with the efficiencies of two other randomization procedures, and it is proved that MRP has an asymptotic efficiency of 100 per cent.

1. Description of the Multistage Randomization Procedure (MRP)

A simple procedure for the randomization of the integers $1, 2, \ldots, n$ consists in (1) writing these integers on cards, one on each, (2) distributing these cards into 10 decks, using random digits $1, 2, \ldots, 9, 0$, (3) repeating step (2) with each deck until single cards or decks of two cards remain, (4) then randomizing each deck of two cards using one random digit and the rule "keep the order if the digit is odd, otherwise reverse the order", and (5) putting all randomized decks successively on top of a result deck. We shall refer to this procedure as the multistage randomization procedure, or MRP.

As an example consider a deck of 15 cards numbered $1, 2, \ldots, 15$, with number 1 on the top card. Using the 15 first digits on row 21 of page 23 of Kendall and Smith (1939), i.e. the digits 4, 9, 8, 7, 9, 8, 3, 0, 2, 8, 2, 9, 9, 4 and 1, we put the first three cards as bottom cards into decks Nos. 4, 9, 8, etc. Thus we get the following 10 "decks":

Deck number	1	2	3	4	5	6	7	8	9	10
Numbers										
on cards									13	
								10	12	
		11		14				6	5	
	15	9	7	1			4	3	2	8

Due to chance two "decks" are empty. Deck No. 1 contains only one card, so this card will be the bottom card of the result deck. Deck No. 2 contains two cards, and these are randomized using the next random digit, which is 3. Since this is odd we keep the order of deck No. 2 and put it on top of the result deck. The latter deck now consists of the cards numbered from top to bottom 11, 9 and 15. On top of these cards we next put deck No. 3, i.e. the card numbered 7. To randomize deck No. 4 we use the next digit which is 4. Hence we reverse the order of deck No. 4 before putting it on top of the result deck.

The next deck which needs to be randomized is No. 8. Using the random digits 6, 1, 1 we get 10 "subdecks", as follows:

Subdeck number	1	2	3	4	5	6	7	8	9	10
Numbers on cards	3									
	6					10				

All of these subdecks except two are empty. We need an extra random digit to randomize the first subdeck. We get the digit 2, and hence reverse the order of subdeck 1 before putting it on the result deck. Then the card numbered 10 is put on the result deck.

Deck No. 9 is distributed into subdecks using the random digits 0, 6, 1, 0, and thus the cards numbered 2 and 13 appear in the tenth subdeck. Since the next random digit is odd the card numbered 2 appears now on top of the result deck. Finally, the card numbered 8 is put on the result deck.

The result deck will now have the 15 cards ordered as follows from top to bottom:

8, 2, 13, 12, 5, 10, 6, 3, 4, 1, 14, 7, 11, 9, 15.

It is advisable to mark the 10 deck numbers with particular cards and to put the subdecks in a separate row along the first set of decks. If n is large, the number of deck-rows needed will sometimes be so large that it is suitable to put aside the first set of decks and then randomize each deck separately. By using a set of edge-punched cards which can be easily ordered, subsets corresponding to the actual number of integers can be taken out; the work of writing numbers on cards does not then have to be repeated each time the procedure is applied.

Methods can be easily constructed that require on the average fewer random digits than MRP, but instead such methods often require more thinking or more computation than MRP.

2. Applications

MRP is particularly useful when randomizing large numbers of objects, but it can also be used when drawing one or more large samples from a finite population. Three applications will now be mentioned.

1. *Randomization within blocks of fractional factorial experiments.* Cochran and Cox (1957) give tables of random permutations of 9 and 16 integers. Some of the fractional factorial experiment designs published by the Statistical Engineering Laboratory of the National Bureau of Standards (1957) and by Connor and Zelen (1959) require, however, blocks with more than 16 units, the largest numbers of units per block being 32, 64 and 128 for designs with factors at two levels, and 27, 81 and 243 for designs with factors at three levels.

2. *Randomization, with replication, in work sampling.* Suppose one wants to randomize the order of k observations on each of m persons, i.e. together $k.m$ observations. This can be done by randomizing km cards consisting of k sets of cards with each set numbered $1, 2, \ldots, m$.

3. *Sampling a large proportion of a finite population.* Suppose a population of 150 objects is given and that a random sample of 50 objects is required. Suppose also that the 50 sample objects do not have to be recorded in a random order. Using a standard method based on 3-digit random numbers at least 150 random digits will be required. Without considerably increasing this number the required sample can be rapidly obtained by means of MRP. Only one of the 10 decks will have to be split: the one containing the fiftieth card. The procedure can be improved. Since the sizes of the first set of decks are independent of the numbers written on the cards, one can make the choice of decks depend on their sizes. In this way one can often reduce the size of the deck containing the fiftieth card.

3. MRP GIVES ALL PERMUTATIONS WITH EQUAL PROBABILITIES

In this section we shall prove that MRP gives each permutation of the n cards with the probability $1/(n!)$.

This statement holds for $n = 1$. On the assumption that it holds for $1, 2, \ldots, n-1$ cards we show that it holds for n cards.

Consider one particular permutation $a_1 a_2 \ldots a_n$ of n cards. Let deck number 1 consist of the first n_1 of these cards, let deck number 2 consist of the next n_2 cards and so on, and let deck number 10 consist of the last n_{10} cards, where the largest n_i $(i = 1, \ldots, 10)$ is less than n. The probability of getting, in step (2) of MRP, this distribution over 10 decks, before randomizing the decks, is p^n, where $p = 1/10$. Since step (2) is repeated if one of the n_i equals n, the conditional probability of the same distribution over decks, given that no deck contains n cards, is $p^n/(1-10p^n)$, or $p^n/(1-p^{n-1})$.

The probability of getting the permutation $a_1 a_2 \ldots a_{n_1}$ of the n_1 cards of deck number 1 is $1/(n_1!)$, since, by assumption, MRP gives equal chances to all permutations of $n-1$ or fewer cards. Similarly the probability of getting the permutation $a_{n_1+1} a_{n_1+2}, \ldots, a_{n_1+n_2}$ of the n_2 cards of deck number 2 is $1/(n_2!)$ and so on. Hence the probability of getting the permutation $a_1 a_2 \ldots a_n$ and n_i cards in the ith deck $(i = 1, \ldots, n)$ in step (2), given that not all n cards will appear in the same deck, is

$$\frac{p^n}{1-p^{n-1}} \cdot \frac{1}{n_1!\, n_2! \ldots n_{10}!}.$$

But the sum of all such probabilities over all possible sets of 10 decks with no n_i equal to n is $(1/n!)$, since the sum over all n_i, with no n_i equal to n, of the multinomial probability

$$\frac{n!}{n_1!\, n_2! \ldots n_{10}!} \cdot p^n$$

equals $1-p^{n-1}$. This completes the proof.

4. THE EXPECTED NUMBER OF RANDOM DIGITS REQUIRED BY MRP

Due to the statistical nature of MRP the number of random digits required to randomize n symbols is itself a random variable. In the present section the expected value of this random variable is considered. This quantity is fairly easy to compute for moderate sizes of n.

We shall denote by $g(n, r)$ the expected value of random numbers required by a generalized MRP for which a number system based on r symbols is used. Since $g(n, r)$ equals n plus the sum of the expected numbers for the r decks of step (2), which are all determined by the same binomial distribution, we get

$$g(n,r) = n + r \sum_{i=0}^{n} \binom{n}{i} \left(\frac{1}{r}\right)^{i} \left(1-\frac{1}{r}\right)^{n-i} g(i,r). \tag{4.1}$$

For our present procedure, where $r = 10$, we write $g(n, 10) = g(n)$. Solving (4.1) for $r = 10$, we get the recursion formula

$$g(n) = \begin{cases} 0 & (n = 0, 1) \\ 1 & (n = 2) \\ \dfrac{n + 10 \sum_{i=2}^{n-1} \binom{n}{i} (1/10)^{i} (9/10)^{n-i} g(i)}{1-(1/10)^{n-1}} & (n \geqq 3). \end{cases} \tag{4.2}$$

Values of $g(n)$ for $n = 2(1)\,25(5)\,50(10)\,100$ are given in Table 1 (overleaf).

5. A MEASURE OF EFFICIENCY FOR RANDOMIZATION PROCEDURES

As a measure of the efficiency of any procedure for randomizing $n\,(>1)$ different symbols by means of random decimal digits such that each permutation will have the same probability of appearing we shall use the quantity E_0 defined as

$$E_0 = \frac{\log_{10}(n!)}{\text{Expected number of random decimal digits required}} \quad (n > 1). \tag{5.1}$$

The choice of E_0 is based on the following considerations. To choose at random one of 10^k arrangements, in restricted randomization, a set of k random digits is necessary and sufficient. To choose one out of the $n!$ possible arrangements of n symbols the number of random digits, K, must be so large that 10^K is not less than $n!$ Thus K cannot be less than $\log_{10}(n!)$. Hence $E_0 \leqq 1$. However, it is possible to construct a randomization procedure, which with unlimited application gives an efficiency arbitrarily close to 1. Suppose x independent randomizations of n symbols are carried out, where x is a large number. Writing $n! = N$ there are together N^x possible results of all these randomizations. We assume that all these results are ordered. Let now y be an integer for which $10^y > N^x$. If sets of y random digits are sampled until a set corresponding to one of the N^x randomizations is obtained then, according to a well-known result in inverse sampling (see, for example, Olds (1940), formula (6), with an infinite population), the expected number of sampled sets is $1/P$, where $P = N/(10^y)$. Hence the expected number of random digits per randomization of n symbols is $y/(Px)$ or $(y/x)\{10^y/(N^x)\}$. Writing A for the latter quantity we now show that $A/\log_{10} N$ can be made arbitrarily close to 1. We first observe that there exists an x-value, say x_0, which is larger than $1/\epsilon$, where ϵ is an arbitrarily small positive number, and a corresponding y-value, say y_0, such that the inequalities

$$0 < y_0 - x_0 \log_{10} N < 1/x_0 < \epsilon \tag{5.2}$$

hold. This follows from Theorems 7.10 and 7.11 of Niven and Zuckerman (1960), if y_0/x_0 is chosen as a convergent, of sufficiently large odd index, of the irrational number $\log_{10} N$. Hence $10^{y_0}/N^{x_0}$ can be made arbitrarily close to 1, and the same holds for $y_0/(x_0 \log_{10} N)$. It follows that $A/\log_{10} N$ can be made arbitrarily close to 1. Thus the efficiency attained in the corresponding x_0 randomizations can be made arbitrarily close to 1. The same holds for x randomizations, if x satisfies the inequalities $kx_0 < x \leqq (k+1)x_0$, provided k is a sufficiently large integer. For in this case the x randomizations can be split up into k sets of x_0 randomizations plus a residual set,

TABLE 1

The efficiencies of the three methods, MRP, ORP and MORP

Number of symbols	*Expected number of random decimal digits required by*			*Efficiency (per cent.) of*		
n	*MRP*	*ORP*	*MORP*	*MRP*	*ORP*	*MORP*
2	1·000	1·000	1·000	30·1	30·1	30·1
3	3·303	4·333	2·111	23·6	18·0	36·9
4	4·610	6·833	3·361	29·9	20·2	41·1
5	6·018	8·833	4·361	34·5	23·5	47·7
6	7·525	10·50	6·028	38·0	27·2	47·4
7	9·127	11·93	7·457	40·6	31·0	49·7
8	10·82	13·18	8·707	42·6	34·9	52·9
9	12·59	14·29	9·818	44·1	38·9	56·6
10	14·45	15·29	10·82	45·4	42·9	60·6
11	16·37	33·47	12·84	46·4	22·7	59·2
12	18·37	50·14	14·92	47·3	17·3	58·2
13	20·42	65·52	17·12	48·0	14·9	57·2
14	22·53	79·81	19·16	48·6	13·7	57·1
15	24·69	93·14	21·38	49·1	13·0	56·7
16	26·90	105·6	23·47	49·5	12·6	56·8
17	29·14	117·4	25·82	49·9	12·4	56·4
18	31·43	128·5	28·04	50·3	12·3	56·4
19	33·74	139·0	30·15	50·6	12·3	56·7
20	36·08	149·0	32·15	51·0	12·3	56·7
21	38·45	158·6	34·53	51·3	12·4	57·1
22	40·84	167·7	36·80	51·5	12·6	57·2
23	43·25	176·4	38·97	51·8	12·7	57·5
24	45·68	184·7	41·06	52·1	12·9	57·9
25	48·12	192·7	43·06	52·3	13·1	58·5
30	60·49	228·5	54·99	53·6	14·2	59·0
35	73·03	258·9	67·04	54·8	15·5	59·7
40	85·67	285·2	80·22	55·9	16·8	59·7
45	98·40	308·5	91·86	57·0	18·2	61·0
50	111·2	329·3	102·3	58·0	19·6	63·0
60	137·2	365·5	138·4	59·7	22·4	59·2
70	163·8	396·1	169·0	61·1	25·3	59·2
80	191·0	422·6	195·5	62·2	28·1	60·8
90	219·0	446·0	219·0	63·1	31·0	63·1
100	247·7	467·0	239·9	63·8	33·8	65·8

and for each of the k sets the efficiency can be made arbitrarily close to 1, while for the residual set the efficiency always can be made positive. We have thus shown that, in a sufficiently large number of randomizations of n symbols, the efficiency can be made arbitrarily close to 1.

6. THE EFFICIENCY OF MRP COMPARED WITH THAT OF ANOTHER SIMPLE RANDOMIZATION PROCEDURE

In this section MRP will be compared with another simple randomization procedure, which can be described as follows. Suppose the number of symbols to be randomized, n, satisfies the inequality

$$10^s < n \leqq 10^{s+1}, \tag{6.1}$$

where s is an integer. The n symbols are numbered $1, 2, \ldots, n$. First $(s+1)$-digit random numbers are sampled until a number $\leqq n$ is obtained (a random number consisting of $s+1$ zeros is read as 10^{s+1}). The symbol with the same number is then chosen as the first symbol of the random arrangement of the n symbols. The remaining $n-1$ symbols are renumbered $1, 2, \ldots, n-1$ (if necessary). Next random numbers are sampled until a number not greater than $n-1$ is obtained, and the corresponding symbol is chosen as the second symbol of the random arrangement, etc. When the number of remaining symbols is greater than 10^{s-1} but not greater than 10^s s-digit random numbers are used. When the number of remaining symbols is greater than 10^{s-2} but not greater than 10^{s-1} $(s-1)$-digit random numbers are used, etc. When two symbols remain only one 1-digit random number is needed, as with MRP. If, for example, $n = 30$ the first 20 symbols are sampled by means of 2-digit random numbers, and the next 9 symbols are sampled by means of 1-digit random numbers.

The sampling and renumbering can be made rapidly if the symbols are represented by, for example, numbered metallic spheres which are placed between two parallel and fixed rods, one of which has a scale on which the number of non-sampled spheres can be read off exactly. When sampling a sphere one uses the numbers on the scale. After sampling a sphere, the remaining ones are "renumbered", i.e. moved together so that the length of the row of spheres is decreased by one unit. We shall refer to this procedure as the one-stage randomization procedure, or ORP. It is developed from the simplest one of the procedures described by Fisher and Yates (1943).

The expected number of random digits required by ORP can be determined using inverse sampling and the formula

$$\sum_{i=1}^{r} 1/i = \log_e r + C + 1/2r - \theta/8r^2, \tag{6.2}$$

where $C = 0{\cdot}5772\ldots$ is Euler's constant and θ is a number between 0 and 1 (cf. Cramér, 1946, section 12.2).

Suppose now that $n = 10^s + m$, where $s \geqq 1$. Then the probability that an $(s+1)$-digit random number is not greater than n is $n/(10^{s+1})$. Hence the expected number of $(s+1)$-digit random numbers needed to sample the first symbol using ORP is $10^{s+1}/n$. Similarly, if $i \leqq m$, the expected number of $(s+1)$-digit random numbers needed to sample the ith symbol using ORP is $10^{s+1}/(10^s + m - i + 1)$. Hence the expected number of random digits needed to sample the first m symbols is

$$(s+1) \sum_{i=1}^{m} \frac{10^{s+1}}{10^s + m - i + 1} = 10^{s+1}(s+1)\left(\sum_{i=1}^{n} (1/i) - \sum_{i=1}^{10^s} (1/i)\right)$$

$$\sim 10^{s+1}(s+1)\left(\log \frac{n}{10^s} + \frac{1}{2n} - \frac{1}{2.10^s}\right). \tag{6.3}$$

In the same way the expected number of random digits needed to sample the next $10^{s+1}-10^s$ symbols is approximately

$$10^s s\left(\log 10+\frac{1}{2.10^s}-\frac{1}{2.10^{s-1}}\right). \tag{6.4}$$

The expected number of random digits required by ORP when randomizing n symbols is given, without approximation, for $n = 2(1)25(5)50(10)100$ in Table 1. In the same table the efficiencies of MRP and ORP are given for the same set of n-values. We find that the efficiency of ORP is much lower than that of MRP for most of these n-values.

7. The Efficiency of MRP compared with That of a Fairly Efficient One-stage Procedure

In this section we consider a modification of ORP, here called MORP, which coincides with one of the procedures suggested by Fisher and Yates (1943), except that the number of digits constituting the random number is not held constant when n is greater than 10. The procedure will be described for $n = 38$. To choose the first symbol of the random arrangement we sample 2-digit random numbers until we get one of the numbers 01–76, 76 being the largest integer multiple of 38 not exceeding 100. Suppose we get the random number 47. This number is divided by 38, giving the residual 9. Then the ninth out of the 38 symbols is chosen. (A remainder equal to 0 is counted as 38 in this case.) When 10 or fewer symbols remain we sample 1-digit random numbers. When, for example, 4 symbols remain we sample 1-digit random numbers until we get one of the numbers 1–8, 8 being the largest integer multiple of 4 not exceeding 10, and so on.

From Table 1 it is seen that MORP gives a higher efficiency than MRP for most of the n-values in the tables. However, MRP is much easier to apply than MORP, at least for large values of n.

8. The Asymptotic Efficiency of MRP

In this section we show that the asymptotic efficiency of MRP, as n goes to ∞, is 100 per cent. For this purpose we first construct a family of upper bounds on $g(n)$.

Consider the function

$$h_1(n) = an^2+bn+c, \tag{8.1}$$

where we choose a, b and c such that $h_1(n) = g(n)$ for $n = 1$, 2 and 3. Using (4.2) we get the equation system

$$\begin{aligned} a+\ b+c &= 0 \\ 4a+2b+c &= 1 \\ 9a+3b+c &= 109/33 \end{aligned} \tag{8.2}$$

which has the solution

$$a = 43/66, \quad b = -63/66, \quad c = 20/66. \tag{8.3}$$

We now prove the inequality

$$h_1(n) \geqq g(n) \quad (n = 0, 1, 2, \ldots). \tag{8.4}$$

Obviously (8.4) holds for $n = 0$, 1, 2 and 3. Assuming that $h_1(j) \geqq g(j)$ for $j = 1, 2, \ldots, (n-1)$, we have, for $n \geqq 3$, using (4.2) and writing p instead of 1/10

$$(1-p^{n-1})(h_1(n)-g(n)) = (1-p^{n-1})\,h_1(n) - n - 10\sum_{i=1}^{n-1}\binom{n}{i}p^i(1-p)^{n-i}g(i) \geqslant A(n),$$

where
$$A(n) = (1-p^{n-1})\,h_1(n) - n - 10\sum_{i=0}^{n}\binom{n}{i}p^i(1-p)^{n-i}(ai^2+bi+c)$$
$$+10p^n h_1(n) + 10c(1-p)^n$$
$$= h_1(n) - n - 10(ap^2n^2 + ap(1-p)\,n + bpn + c) + 10c(1-p)^n$$
$$= (1-p)an^2 - (1+(1-p)\,a)\,n - 9c + 10c(1-p)^n$$

or
$$A(n) = 0{\cdot}9an^2 - (1+0{\cdot}9a)\,n - 9c + 10(0{\cdot}9^n)\,c.$$

Now $A(3) = 0$. Further, for real x,

$$\frac{dA(x)}{dx} = 0{\cdot}9a(2x-1) - 1 + 10c(0{\cdot}9^x)\log_e 0{\cdot}9.$$

Since $\log_e y < y-1$ for $y>0$ and $y \neq 1$ (see, for example, Hardy *et al.*, 1934, p. 106), we have that $\log_e(10/9) < 9^{-1}$, so that, for $x > 1$,

$$0{\cdot}9^x \log_e 0{\cdot}9 > -0{\cdot}9^x(9^{-1}) > -10^{-1}.$$

Hence
$$\frac{dA(x)}{dx} > 0{\cdot}9a(2x-1) - 1 - c \quad \text{for} \quad x > 1,$$

and thus
$$\frac{dA(x)}{dx} > 0 \quad \text{for} \quad x \geqq 3.$$

It follows that $h_1(n) \geqq g(n)$ for integer $n \geqq 3$, which completes the proof of (8.4).

We shall now prove that, for $k = 1, 2, \ldots$ and $n = 0, 1, 2, \ldots$,

$$g(n) \leqq h_k(n) = a_k n^2 + b_k n + c_k, \tag{8.5}$$

where $\quad a_k = 10^{-k+1}a, \quad b_k = k - 1 + (1-10^{-k+1})\,a + b \quad$ and $\quad c_k = 10^{k-1}c. \qquad$ (8.6)

We know that (8.5) holds for $k = 1$. Assuming that (8.5) holds with $k-1$ instead of k we have, for $n \geqq 3$,

$$g(n) \leqq n + 10\sum_{i=0}^{n}\binom{n}{i}p^i(1-p)^{n-i}h_{k-1}(i). \tag{8.7}$$

We now verify that the right member of (8.7) can be denoted by $h_k(n)$. In fact the coefficient of n^2 in the right member of (8.7) is

$$10p^2 a_{k-1} = 10^{-1}a_{k-1} = 10^{-1}(10^{-k+2}a) = 10^{-k+1} = a_k.$$

Similarly b_k and c_k are verified. Thus (8.5) holds for $n = 3, 4, \ldots$, and $k = 1, 2, \ldots$. Using (8.3) we finally verify that (8.5) holds for $n = 0, 1$ and 2, and $k = 1, 2, \ldots$.

Now the efficiency of MRP is

$$E_0 = \frac{\log_{10}(n!)}{g(n)}. \tag{8.8}$$

Using Stirling's formula and (8.5) and (8.6) we get

$$E_0 > \frac{(n+\frac{1}{2})\log_{10} n - n\log_{10} e}{h_k(n)}. \tag{8.9}$$

For each n we now choose k such that

$$\log_{10} n \leqq k < \log_{10} n + 1. \tag{8.10}$$

Hence $$h_k(n) < 10a(n-1) + n(a+b+\log_{10} n) + cn.$$

Thus E_0 is larger than a quantity which has the limiting value 1, as $n \to \infty$.

Hence MRP has the limiting efficiency 100 per cent. It is easy to verify that neither ORP nor MORP is asymptotically efficient.

9. Note on a Procedure suggested by Rao

The author developed MRP in 1958. The present paper was almost completed when the author learnt that Rao (1961) had presented a randomization procedure of which MRP is essentially a particular case. Rao, whose method uses pen and paper and who does not mention the use of cards, recommends a one-way classification for $n \leqq 10$, a two-way classification for $10 < n \leqq 100$, and so on. A two-way classification corresponds to 100 decks, and requires that two random digits are chosen at a time at the step corresponding to distributing cards into decks. The efficiency of this procedure is somewhat lower than that of MRP for n-values between 10 and 100, but the difference decreases with increasing n in this interval. Thus the expected number of random digits required by Rao's procedure for $n = 11$, 50 and 100 is 22·55, 112·4 and 247·7, respectively.

10. A Modification of MRP

All methods mentioned in the previous sections have rather low efficiencies for small n, except MORP, and require one random decimal digit to randomize 2 symbols. A simple way of improving MRP, without at the same time completely destroying its simplicity, is to transform random decimal digits into random bits, i.e. random binary digits. The following procedure is both simple and fairly efficient. Draw a pair of random decimal digits. When a random decimal digit equals one of the numbers 0–7 we get 3 random bits from it, and when it equals 8 or 9 we get 1 random bit from it. Each time the pair gives together 4 random bits in this way we get a fifth random bit by observing which of the decimal digits of the pair gives 3 bits. Hence the expected number of random bits obtained from a pair of random decimal digits equals $(0{\cdot}64)(6)+(0{\cdot}32)(5)+(0{\cdot}04)(2) = 5{\cdot}52$, or 2·76 per each decimal digit. Thus in unlimited application of this method $1/2{\cdot}76 = 0{\cdot}36232$ random decimal digits are required, on the average, to randomize two symbols. This gives an efficiency of 83·1 per cent. Using a supply of random bits, obtained in this way, for step (4) of MRP (cf. section 1), the expected number of random decimal digits required will be 5–10 per cent. lower for moderate sizes of n, as compared with MRP. Thus for $n = 3$, 12 and 30 the gain will be about 5, 10 and 6 per cent., respectively.

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