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AN EXCEPTIONAL TALENT FOR CALCULATIVE THINKING

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This study explores the highly exceptional 'lightning calculation' of a distinguished mathematician who has considerable understanding of his own calculative thinking. Each calculation is a temporally co-ordinated, rapidly flexible onleading which is both unitary and complex. Biographically, it derives from prolonged and intensive practice fostered by circumstances in his upbringing and made possible by a large cognitive capacity which also manifests itself in other forms of intellectual achievement. During calculation, there are 'leaps' of varying compass; there is also notable absence of sensory-type awareness. Ongoing proceeds through apprehending multiple attributes of the presented problem, deciding on some convenient and often ingenious calculative plan, and rhythmically implementing this plan while carrying through opportunistic telescoping and verifying of the ongoing activity.

I. INTRODUCTION

Prof. Alexander Craig Aitken, F.R.S., was born in New Zealand in 1895. He is a mathematician of recognized distinction who, since 1946, has occupied the Chair of Mathematics in Edinburgh University. He also calculates mentally with a skill which possibly exceeds that of any other person for whom precise authenticated records exist—samples of such records are to be found in Bidder (1856), Scripture (1891), Binet (1894), Mitchell (1907), and Jakobsson (1944). Although Prof. Aitken's calculative skill has gained him a reputation both in Edinburgh and beyond, he rightly values it less than his more high-level, complexly creative intellectual accomplishments. It is a professionally useful side-line which he is reluctant to sensationalize, and only one of his published papers refers extensively to it (Aitken, 1954). This paper contains a talk given to the Society of Engineers on the art of mental calculation. It also contains reports of calculations proposed by the audience during the talk. One of these reports may be quoted to indicate the high order of mental skill under consideration.

'Dr [H. G.] Taylor here asked for the squares of the three-digit numbers 251, 299, 413, 568, 596, 777 and 983, each of which was correctly given almost instantaneously, 568 and 777 taking a little longer. Dr Taylor then proposed the four-digit numbers 3189 and 6371; in each case the square was given in about 5 sec., the lecturer making a momentary error and correcting it in the first case...Dr Taylor here proposed [for square root] several of the previous numbers, namely, 251, 299, 413, 596, 777. In each case the square root was given in 2 or 3 sec. to five significant digits, with the remark that for 299 and 596 the last digit might be in excess, which it was. Dr Taylor then proposed [for square root] the four-digit numbers 3189 and 8765. In each case, the result was quickly given to five digits' (p. 298).

The aim of the present study is simply to consider one relatively restricted class of Prof. Aitken's calculative thinking and try to discover as much about it as possible: this class of thinking is decimalizing, that is, transforming a numerical fraction into a decimal. No attempt is made to advance any particular theory of thinking nor to give a rounded psychological portrait of this thinker whose intellectual achievements are remarkable even outwith calculation and mathematics. The aim is naturalistic description of a limited class of highly developed thinking activity. The study was conducted by interviews and correspondence over a period of a year. Numerical problems were presented to Prof. Aitken and his solving of them was discussed at length. Some interviews were brief, a matter of minutes; others lasted an hour or more; some were informal; some were tape-recorded. It is unfortunate that the tape-recorded interviews occurred at a time when Prof. Aitken was admittedly tired and clearly not in best form. The following extract from a tape-recording illustrates the rapid accuracy of decimalizing even at a time when he is not 'in form'.

Problem 1. Decimalize 3 over 408. Pause of 6 sec. during which the data are slowly repeated. 'Point 0, 0, 7, 3, 5, 2, 9, 4, 1, 1, 7, that's as far as I can go.' These digits are spoken at a uniform rate of one every second. He is asked if he has reached the repeating period. 'No, I could go on to it, I think, now. Yes, 1, 1, 7, 6, 4, 7, 0, 5, 8, 8, 2, 3, now that must be it. 3, 5, 2, 9, and so on. The repeating period is sixteen digits.' Again, these digits are produced at a steady rate of one every second. It was later remarked that this was a clumsy problem to present, for it was only 1/136.

Problem 2. Decimalize 2 over 63. Pause of 4 sec. during which the problem is repeated. 'Point 0, 3, 1, 7, 4, 6, then the whole thing repeats.' These digits are spoken at a steady pace of two per second.

Problem 3. Decimalize 3 over 78. Pause of 4 sec. during which the problem is repeated. 'Point 0, 3, 8, 4, 6, 1, 5, and those last six digits, from the 3 to the 5, repeat.' Those digits are given at a steady rate of two per second. Later, it was remarked that this problem was simply 1/26.

Problem 4. Decimalize 1 over 752. Pause of 9 sec. during which the problem is repeated. 'Point 0, 0, 1, 3, 2, 9, 7, 8, 7, 2, 3, 4, that's as far as I can get.' These digits are spoken at a steady pace of about one per second with a speeding up over the last four digits.

Problem 5. Decimalize 1 over 57. Pause of 3 sec. during which the data are repeated. 'Point 0, 1, 7, 5, 4, 3, 8, 6, and I can't carry it any further.' He is asked why he does not take it further. 'Well, it's just tiredness just now. If I were fresh, I could carry that further, I think. For instance, I said 6 and I know now that it's 59. You see, I was beginning to approximate. It's 43859, I know that much now.'

These and similar tape-recordings reveal some features of the calculating. Most obviously and impressively, they show the speed of attaining solution and the ease with which an interrupted calculation can be resumed after a lapse of time (when the thinker says he cannot carry an answer further, he does not necessarily mean that he is literally unable to continue but, rather, that he feels it unnecessary to proceed further). These records show, too, the characteristic initial pause and the relatively uniform pace of speaking the answer; also that, according to the nature of the problem, there are variations both in length of pause and in pace of answering. Other recordings strongly indicate that pause and pace are affected by warming-up and fatigue but that, apart from this, pace is a stable characteristic of answering any particular problem, i.e. when the same problem is presented at different times and worked by the same method, the pace of answering is much the same from one occasion to the next. When the thinker reports that a problem was especially difficult to solve, the pace of answering is usually found to have been irregular.

Apart from the above points, these records of calculation in progress reveal little else about the nature of the thinking involved; particularly as the thinker uses no gestures but sits, relaxed and still, while calculating. So it is necessary to ask for introspective reports and explanatory comments. Fortunately, these introspections and elaborations are highly informative. For example, problem 4 was done by taking 752 as 8 times 94, then doing a 94th and dividing the outcome by 8, running the two divisions together. Here is an example of two related commentaries which were submitted in writing.

Problem 6. Decimalize 1/851. 'The instant observation was that 851 is 23 times 37. I use this fact as follows. 1/37 is 0·027027027027...and so on repeated. This I divide mentally by 23. [23 into 0·027 is 0·001 with remainder 4.] In a flash I can see that 23 into 4027 is 175 with remainder 2, and into 2027 is 88 with remainder 3, and into 3027 is 131 with remainder 14, and even into 14,027 is 609 with remainder 20. And so on like that. Also before I ever start this, I know how far it is necessary to go in this manner before reaching the end of the recurring period: for 1/37 recurs at three places, 1/23 recurs at twenty-two places, the lowest common multiple of 3 and 22 is 66, whence I know that there is a recurring period of 66 places.'

Six weeks later, he reports on the decimalizing of other fractions which have 851 as denominator. 'Certainly 15/851 and 17/851 would produce different reactions. For the first, I should have started at once to divide 0·405405405...by 23, three digits at a time. [This is the same basic procedure as before. 15/37 is 0·405405 repeating.] The wiser course with 17/851 would be to do the corresponding thing upon 0·459459459459... [this is 17/37]; but the fact that 17 into 851 is so very nearly 50 would make me want to correct 0·02 by subtracting 1/851 of it, in my own way of rapid compensation. In fact I should have 0·0199765 in a flash; but could not continue so easily, since dividing by 851 and adjusting is not so easy as dividing straight on by 23. I spoke before about not changing horses in mid-stream. Here is a case where the choice would be made at once; I should in fact go ahead with division of 0·459459459... by 23, perhaps deciding to divide two digits at a time instead of three.'

In brief, Prof. Aitken was, for the purposes of this study, posed two kinds of problem. The primary problem was calculative, that is, thinking was directed to attaining a numerical solution. The secondary problem was introspective and explanatory, that is, thinking was directed to attaining a description of primary solving activities. An absolutely complete description was never possible: in part because much calculative activity was unavailable to self-observation, in part because much that was available could not be put readily into words. Nevertheless, these introspections and elaborations were far from meagre and appeared markedly self-consistent. In contrast to many 'lightning calculators', this thinker is a highly accomplished and creative man with wide general interests, he is accustomed to communicating with other people about abstract matters, and is professionally interested in studying mathematical and calculative thinking. So after completing a calculation, he is able to describe its main outlines and some of his subjective experiences; and, if need be, he can supplement this by re-working all or part of the problem, or by working a similar problem. Thus, throughout this study, heavy reliance was placed on what the thinker had to say about his own thinking.

In considering these self-descriptions, some distinction must be made between formal and experiential description. The above report on problem 6 exemplifies a formal report: it spells out a play by play sequence of events in terms of a socially shared language. It is informative, but it is also incomplete and even misleading, for the relation between it and the actual calculation is rather like the relation between verbal instructions for doing a conjuring trick or tying a shoe-lace and the deft execution of these activities. Experiential reports refer to the thinker's experiences during calculation, e.g. to visual imaging if any. They are more difficult for this thinker to achieve; they are also more difficult to interpret. The next section emphasizes formal self-descriptions, and experiential reports will be considered later.

II. Formal characteristics of mental calculating

Each calculation proceeds, from start to finish, as a temporally extended, ongoing unity: it is a progressive unfolding, opening out, *leading on*. From the moment of encountering the problem, there is continuous, dynamic extending of the numerical data. It cannot be over-emphasized that this onleading is a unitary activity without distinct break—albeit a unity which is complex, extended in time, and markedly rhythmic. In the interests of analysis, it is possible to consider now this, now that facet. It is especially convenient to isolate three main focal regions; but it must be stressed that these foci are artificially abstracted from a many-sided onleading in which the thinking of each moment is intimately bound to what has preceded and what is predetermined to follow.

In so far as these focal regions can be discussed as separate phases of the dynamic onleading, they typically occur in the following sequence. (1) Apprehending the problem. Here the class of problem (e.g. decimalizing) and the numerical data (e.g. 1/43) conspire to direct the initial mode of onlead. (2) Deciding the calculative plan. This phase emerges directly from the first and merely carries apprehending further by setting the main course through which leading on will continue. (3) Executing the plan. This leads on further still.

Throughout these three phases, there are some striking general features. There is intimate familiarity with the constraints of calculative language: a rapid ease of converting one set of numerical attributes into other sets; a large repertoire of readily initiated calculative plans along with an appreciation of their implications for the problem in hand. There is large cognitive capacity: ability to carry through a multiplicity of interrelated activities in a short space of time, to act complexly and still hold pertinent data securely ready for use when required. There is the use of rhythm to sustain and co-ordinate this complex onleading. Lastly, there is what might be called sense of propriety: a flexible, opportunistic deploying of resources, an ever-present feeling for what is most fit with regard to the immediate mode of leading on, the future outcomes of super-ordinate plans, the rhythm of thinking; all this in relation both to the data and to his own capabilities.

The three foci mentioned above will now be discussed separately in more detail.

(1) *First phase: apprehending the data*

A number is apprehended as a multiplicity of numerical attributes and, so to speak, as bristling with signalling properties. This apprehending is immediate, simultaneous, and often autonomous. Regarding immediacy, the attributes issue from the thinker's activities, that is, his perceiving or interpreting of the data; yet there is no awareness of activity any more than most people are aware of activity in, say, seeing two objects as being at different distances from them. Regarding simultaneity, these attributes are apprehended all at once. A person glimpses a short, fat, swarthy, grinning man, and apprehends these attributes simultaneously. A Japanese paper flower is placed in water, and unfolds in several directions, continuing through a variety of co-developments. So it is, for this thinker, with attributes of a presented number.

This simultaneous, immediate apprehending of numerical attributes is often autonomous in that no specific preparation is necessary. For example, on one occasion the thinker heard the year 1961 mentioned, and apprehended this as 37 times 53, *and* 44 squared plus 5 squared, *and* 40 squared plus 19 squared. He does not have to set himself to apprehend numbers thus; rather he must set himself to prevent such apprehending. 'If I go for a walk and if a motor car passes and it has the registration number 731, I cannot but observe that that is 17 times 43. But as far as possible, I shut that off because it interferes with thought about other matters. And after one or two numbers like that have been factorized, I am conditioned against it for the rest of my walk.' He sometimes finds himself squaring numbers which he sees, for example, on the lapels of bus conductors: 'this isn't deliberate, I just can't help it'. Such autonomous activity was certainly absent before the age of fourteen years and exists now as a result of long persistent curiosity about numbers. 'It is a good exercise to ask oneself the question: what can be said about this number? Where does it occur in mathematics, and in what context? What properties has it? For instance, is it a prime of the form $4n+1$, and so expressible as the sum of two squares in one way only? Is it the numerator of a Bernoullian number, or one occurring in some continued fraction? And so on. Sometimes a number has almost no properties at all, like 811, and sometimes a number, like 41, is deeply involved in many theorems that you know.' He has pursued this sort of exercise for half a century, deliberately and enthusiastically in the earlier years and later out of habit and the circumstances of his mathematical occupation.

The readiness with which a presented number leads on to numerical properties is illustrated in the following excerpt from a tape-recording. For each number, the task was to state whether it is prime or, if not, to state its factorization. In each case, there was no clearly perceptible pause between question and answer, and certainly no pause long enough to be measured by stop-watch.

'*Q*. 963. *A*. Nine goes into that, so it's too simple to bother with. *Q*. 386. *A*. Twice 193: don't have any even ones for they factorize at once. *Q*. 113. *A*. Prime. *Q*. 719. *A*. Prime. *Q*. 533. *A*. 13 times 41. *Q*. 391. *A*. 17 times 23. *Q*. 871. *A*. 13 times 67. *Q*. 1211. *A*. 7 times 173. *Q*. 313. *A*. Prime. *Q*. 417. *A*. 3 times 139. *Q*. 529. *A*. 23 squared: first time [in the entire session of some 30 min.] you've struck a square.'

Although numbers are cues for cognitive action which is often autonomous, it does not follow that apprehending is uninfluenced by context. The way a number leads on varies with the task and with the numbers presented alongside it. Consider the presented number 851 (see problem 6 above). When asked to decimalize 1/851, the immediately dominant attribute is the factorization 23 times 37. When asked to extract the square root of 851, the immediate lead on is that 851 is 29 squared plus 10. When asked to decimalize 17/851, the immediately dominant property is that this fraction is very nearly 0·02. When asked to memorize 8-5-1 as part of a memory span task, there is no numerical leading on: in this situation the separate digits are memorized in a plain, raw sequence.

In short, almost any presented number is rich in meaning, immediately leads on to a constellation of numerical attributes. Some numbers have more attributes than others and the attributes which are apprehended vary with the context—just as, for most people, the same word is apprehended in different ways according to its verbal and situational context.

(2) *Second phase: deciding the calculative plan*

Anyone who is able, say, to decimalize 1/23 plans his calculating in the sense of initiating a schematic and predetermined sequence of activities which, if correctly executed, lead to the answer. But most adults can readily initiate only one plan, namely, straight long division. By contrast, this thinker can readily initiate any of several plans. His repertoire of calculative plans is large, and the same problem can be solved in different ways—although for any particular problem, some of these plans are more 'convenient' than others. Deciding the plan is the key step of the entire performance for it determines the main way in which thinking leads on from the presented data. The following quotation (Aitken, 1954, p. 300) indicates some of the plans which *could* be used with 23 as denominator; it also outlines a modified method of short division which will be referred to later.

'One can divide by a number like 59, or 79, or 109, or 599, and so on, by *short* division. Take, for example, 1/59, which is nearly 1/60. Set out division thus

6 | 1·0 1 6 9 4 9 1 5 2 ...
0·0 1 6 9 4 9 1 5 2 5 ...

Here we have the decimal for 1/59, obtained by dividing 1 by 60; as we obtain each digit we merely enter it in the dividend, *one place later*, and continue with the division.

As another example, consider 5/23. Write it as 15/69. Then proceed

7 | 15·2 1 7 3 9 1 3 0 ...
0·2 1 7 3 9 1 3 0 4 ...

In fact 5/23 = 0·2173913043478260869565, a recurring decimal with a period of 22 digits. One could equally well have written it as 65/299, then carrying out division by 3, two digits at a time, and entering in the dividend *two places* further along.' [Notice the rationale of this method. It replaces the required divisor, say 199, by a more convenient divisor, i.e. 200. This replacement is then compensated for by continuing to increase the value of the dividend by a two-hundredth of itself. The greater convenience of the substitute divisor is even more evident when it is remembered that dividing by 200 is the same as dividing by 2 with compensatory precautions concerning the decimal place.]

'There are other possibilities. For example, the mental calculator is, or should be, very familiar with the factorization of numbers; he should know not merely that 23 times 13 is 299, but that 23 times 87 is 2001. For example 5/23 is equal to 435/2001; and if we note that 435 is the same as 434·99999 ..., we have another method, in which, as we obtain the digits, we *subtract* them from the dividend, so many places later. Thus in the present case

2 | 434 782 608 695 652 ...
217 391 304 347 ...

For example, 217 from 999 gives 782, which we then divide by 2, obtaining 391; this, subtracted from 999, gives 608; and so on.

'My aim has been to demonstrate, in these various rather simple examples, some part of the repertoire, the armoury of resource upon which a mental calculator may draw, and in regard to the choice of which he must make instantaneous decisions, and keep to them.'

Deciding the calculative plan occurs in that initial pause which follows the presentation of all but the very simplest problems. It can happen that, after onleading has begun by one plan, the thinker is suddenly aware of another and more convenient plan which he must prevent himself from implementing. However, the plan decided on is, usually, the plan which is later described as being most convenient for him to use with the particular problem concerned. It is in this deciding that the thinking is perhaps most subtle and opportunistic: it is this phase of onlead for which a prescription is most difficult to write. With problems as simple as those being considered in this study, there is little if any conscious awareness of comparing the utilities of alternative plans. Often there is a feeling of conflict and uncertainty, a non-detailed awareness of competing modes of onlead: but the resolution of this conflict does not involve any conscious provisional working out of this or that plan. The decision can usually be justified after the event but little of this justification seems to enter consciously into the making of the decision. 'I have never thought of assigning a method for "ranking" the convenience of one method as against another. Sometimes, when I have embarked on a calculation and a side-flash of a better method occurs, I think I see that it seems better just because it would have taken fewer steps, would perhaps get me twice as many figures by a more rapid process. But apart from this, I think I see at a glance and without going into deep detail that some method will almost certainly be better than some other.'

It is noteworthy that the number of feasible, as opposed to theoretically possible, plans is greatest with problems of intermediate complexity, more specifically, when the divisor is a two-digit number. With a one-digit divisor, plans other than straight division would introduce unnecessary calculative complexity. With many three-digit and almost all four-digit divisors, plans other than straight division would be too complex to carry through readily as a unit: indeed straight division itself may become laborious to the point of being not worth while attempting mentally.

(3) *Third phase: executing the calculative plan*

Within this schematically predetermined sequence of activities, there is always alertness for alternative ways of executing subsequences, for telescoping and short-cutting chains of calculation, and for apprehending verifying attributes of the on-going thinking. All this is evident in the following introspective report (submitted in writing) on a decimalization done by the method of modified short division mentioned in the preceding section.

Problem 7. Decimalize 1/43. 'As you know, I seize at once on a useful property. In this case, that 43 by 93 is 3999, one less than 4000. At once I begin to divide 93 by 4000, entering the answer at the proper place and continuing with the division. Therefore, I have got 0·0232558.... You will notice that, not bothering further with the position of the decimal point, I have divided 93 by 4, getting 2325; I am adding to this one 4000th of itself—strictly speaking one 3999th, but by the time the little increment has been tacked on, it is indeed one 4000th of what then has been set down. Now I said to you that, while this proceeds, I have flashes from the side, small extraneous checks, verifications, and even hints for telescoping or simplifying. One of these occurs almost immediately above. Note the 558. I instantly observe that it is 6 times 93. Excellent check on my accuracy so far. And as I take leave of it, with a 'glance' (not visual), I am dividing by 4000 (that is, by 4 with proper safeguards as to position) and getting 13953488372.... But again I note in the same way the 837, 9 times 93, checking again as I fly along. Also the 372, 4 times 93, telling me that I am almost there because it is *four* times, and 4 is my divisor. And indeed...8372093 concludes the period. I am back at my first dividend, *with no remainder*, and so everything will now recur.

'But in actual performance, the answer runs with absolute uniformity. The flashes of recognition and reassurance, indicated by asterisks, pass by like flashes of electric bulbs and cause no distraction whatever. 0·023255$\overset{*}{8}$139$\overset{*}{5}$34883$\overset{*}{7}\overset{*}{2}$093. Well, of course, I could have memorized this decimal. And perhaps it is memory as much as calculation. But the calculation is just as fast as if it were pure memory, and the two intertwine indistinguishably.'

That verification activity is a habitual component of executing the plan is evidenced again by the following report, this time on a simple multiplication.

Problem 8. Multiply 123 by 456. 'I see at once that 123 times 45 is 5535 and that 123 times 6 is 738; I hardly have to think. Then 5535 plus 738 gives 56,088. [Note: the 'location' of 5535 is here adjusted so that, in effect, it is 55,350. This thinker does not 'burden the mind with zeros'.] Even at the moment of registering 56,088, I have checked it by dividing by 8, so

7011, and this by 9 gives 779. I recognize 779 as 41 by 19. And 41 by 3 is 123, while 19 by 24 is 456. A check, you see; and it passes by in about 1 sec. [Note: this check breaks the answer down into the originally given numbers by a route which does not merely retrace the method of attaining the answer.]

Perhaps the most outstanding single feature of this thinker is that he leads on so rapidly through such a multiplicity of precisely constrained activities without losing grip on the total calculative ensemble. This large cognitive capacity is clearly evidenced by the above report on problem 7. Quite apart from the running 'side-flashes', this decimalization involves what most people would regard as two distinct, albeit interwoven, chains of calculation. Yet for this thinker, the whole activity leads on as a mobile diversified unity; at most, he experiences one sequence running alongside or underneath another. Thus in so far as distinct calculative sequences are experienced in the modified method of short division, they are experienced as being literally concurrent. However, in more complex calculations there is awareness of alternation. For example, in leading on from the fraction 1/752 (problem 4), a 94th is run and an eighth run on the dividend: here there is distinct awareness of two calculative themes. This explicitly experienced alternating is evidenced in the following transcript of a tape-recording.

Problem 9. Decimalize 1/697. Pause of 5 sec. 'Point 0, 0, 1, 4, I can't get it any further.' These digits are spoken slowly and irregularly, their speaking, from 'point' to 'four', requiring 12 sec. [The method was straight long-division.] 'You've given me very difficult ones. Those with three digits, I must say, are very hard. I didn't expect them or I might have worked up a wee bit of practice. Having said it, I see a trick for getting it alright. I noticed that 697 factorizes, that's all. You see, you asked me for 697 and that is 17 times 41. Therefore, I'll try again and say: point, 0, 0, 1, 4, 3, 4, 7, 2, that's as far as I can go on that one.' These digits are spoken at a steady rate of one per second. 'I did that by a different method. I mentally worked out a 41st and divided it by 17 at the same time. I did two things. That's severe by the way. A double process, it's very severe. You see you've got to run one decimal and divide it at the same time by something else. You've got to alternate back and forward. I've got to be aware that a 41st is point, 0, 2, 4, 3, 9, and be dividing that by 17 along. So I must be aware of a train and an operation on that train. And the numbers have got to keep coming steadily in. And also, I've not got to lose the clue, I musn't jump one. That's highly complicated.'

Later, in writing, Prof. Aitken commented on this calculating as follows. 'I am really disappointed at my showing here. I should have been able to keep 1/41 in mind as 024390243902439... and divided by 17 much further than I did. It would be a long time before the repeating period was reached, because 1/41 repeats at five digits, 1/17 at sixteen digits, and the L.C.M. of 5 and 16 is 80, so it would be an 80-figure period.' [Note: it is more economical to divide a 41st, repeating at five digits, by 17 than to divide a 17th, repeating at sixteen digits, by 41.]

In this example, the thinker is clearly aware of alternating between two calculative chains. Furthermore, each chain can, after being in abeyance, be picked up correctly without hesitation. This immediate memory capacity is referred to in Aitken (1954, p. 305). 'Prof. Aitken replied that he was able to put aside in storage for a future occasion a result that had already been obtained. He knew that he would be able to bring it out correctly.... He thought this ability to put an answer in storage was what distinguished the calculator from what might be called the man in the street. The man in the street forgot the stages in between.' These remarks are substantiated by the following report based on a tape-recording.

Problem 10. Decimalize 4/47 . Pause of 4 sec. during which 'four over forty-seven' is repeated. 'Point, 0, 8, 5, 1, 0, 6, 3, 8, 2, 9, 7, 8, 7, 2, 3, 4, 0, 4, 2, 5, 5, 3, 1, 9, 1, 4, that's about as far as I can carry it.' These twenty-six digits are spoken at a fairly steady rate of one every $\frac{3}{4}$ sec. He is asked whether he has reached the end of the repeating period and, for 52 sec., there is discussion in which he explains that the repeating point might have come after 23 places but, since it had not, it would certainly be at 46 places. Having explained why this should be, he pauses for 5 sec., then begins to complete the decimalization, speaking the digits at a steady rate of one every $\frac{3}{4}$ sec. '8, 9, 3, 6, 1, 7, 0, 2, 1, 2, 7, 6, 5, 9, 5, 7, 4, 4, 6, 8, now's the repeating point. It starts again at 0, 8, 5; so if that is 46 places, I'm right.'

Subsequent discussion reveals that the main plan was to transform 4/47 into 68/799, and then short divide 8 into 68 with the modifications described already. Leading on continued according to this plan until twenty-six digits had been given. At this juncture, it was evident to the thinker that he would have to produce a total of forty-six digits before completing the repeating period; also at this juncture, he noticed that further leading on could be done by an alternative plan. 'And here I stopped, having noticed that 1914 is 3 times 638 which occurred earlier. For a moment I thought I could recover 638297 and so on, and multiply all this by three at the same time as recovering it. But clearly, this was too ambitious and I halted. Later, I returned to my original method based on 47 times 17 equals 799.' Asked if he had any trouble in picking up this interrupted calculation, he replied: 'I know exactly where I was. I always know that. I never have any difficulty in finding the place.'

(4) *Other classes of calculation.* Aitken (1954) outlines some of his calculative plans for obtaining square roots, cube roots, squares, cubes, and factorizations. And it is these plans which seem to constitute the main difference between these calculations and the decimalizing performances already described. The following report (submitted in writing) illustrates the psychological similarity between the activity of leading on to a square root and that of leading on to a decimalization.

Problem 11. Extract the square root of 851. 'I at once perceive that 841 is 29 squared. So 29 is a good first approximation. At once I have noted the remainder 10 and halved it (by my rule) and noted mentally that 5/29 is 0·172.... At once, I risk 29·172 as an answer which is almost certainly correct to five significant digits. But already I am off on another track, because 29·172 is nearly $29\frac{1}{6}$, that is 700/24. And almost before having formulated the procedure in a rational manner, I have divided 851 by 700/24, that is, multiplied it by 24/700. So I feel (rather than see or hear) 20,424/700. But then some experience tells me that 700 times $29\frac{1}{6}$ is 20,416·66666.... Averaging at speed 20,424 and 20,416·66666..., getting 20,420·33333, dividing by 700, and placing the decimal point in the proper place—and all of this in one continuous follow-through like a good golf stroke—I have 29·17190476.... And I will trust 29·171904 because already, with my grasp on this, I have moved on to further considerations. I have seen that 20,420·33333... differs from either 20,424 or 20,416·66666... by about one part in 5600 or slightly less; I have mentally squared and doubled the result and got roughly 62,500,000 which I choose in round numbers because I pull out from the repertoire the fact that 1/16 is 0·0625. Whence I see in a flash that 1/62,500,000 of my previous answer is 0·000000466 or so. This very fine correction, subtracted from my previous, gives 29·17190429 say. And I am satisfied to go no further. I count on doing all of these steps in a total time of less than 15 sec. This example, by the way, was not a 'gift'. Some examples are. For example, the square root of 5187 was once proposed. Well, 72 squared is 5184. Therefore it was a "gift" and, in 1 sec., I had 72·02083 which, by my second approximation, I sharpened at once to 72·020833.'

One final brief report may be added, concerning a large multiplication problem presented to Prof. Aitken some years ago by his own children.

Problem 12. Multiply 987,654,321 by 123,456,789. 'I saw in a flash that 987,654,321 by 81 equals 80,000,000,001; and so I multiplied 123,456,789 by this, a simple matter, and divided the answer by 81. *Answer*: 121,932,631,112,635,269. The whole thing could hardly have taken more than half a minute.'

III. The development of calculative skill

Prof. Aitken's calculative skill may well have been genetically predisposed, for there is a family history of arithmetical accomplishment (see Aitken, 1954, p. 307).

In large part, however, it is the outcome of environmental circumstances and especially of a prolonged and intensive practice at mental calculating which he began at the age of thirteen years. He was not a child prodigy. 'Arithmetic in primary school, since I recall hardly anything about it, must simply have bored me. Possibly, I wasn't taught well. Maybe I simply accepted what the teacher said and did it. When I went up to secondary school, with a scholarship from primary school, I was disappointed at my arithmetic mark. I found that it was only 143 out of 200. I lost 57 marks and if I'd had these I'd have been very high up. As it was, I got a scholarship in spite of that.' So he was not an outstanding arithmetician in primary school and, in his first year at secondary school, he did not win the arithmetic prize. His interest in arithmetic first quickened by an incident during an algebra lesson. 'The master chanced to say that you can use this factorization to square a number: $a^2 - b^2 = (a+b)(a-b)$. Suppose you had 47—that was his example—he said you could take b as 3. So $(a+b)$ is 50 and $(a-b)$ is 44, which you can multiply together to give 2200. Then the square of b is 9 and so, boys, he said, 47 squared is 2209. Well, from that moment, that was the light, and I never went back. I went straight home and practised and found that this reacted on every other branch of mathematics. I found such a freedom. I well remember the stage when I was able to square numbers up to 300 and thought—now that is something! But I was to go far beyond that in future years. And so from the age of what might be $13\frac{1}{2}$ years, up to $17\frac{1}{2}$ when I left that school, I underwent what can only be described as a mental Yoga. I tried harder and harder things until, in the end, I was so good at arithmetic that the master didn't allow me to do arithmetic.'

This intensive practice in calculation (strictly speaking, in mental algebra) did not, however, restrict interest in other matters and, through this, result in a one-sided, idiot savant type of specialization. Rather, it fostered enjoyment in intellectual accomplishments of all sorts. 'Arithmetic wasn't my only interest by any means. I was interested in literature just as much, Latin, French, English. But this great freedom suddenly encouraged me to think I had a memory and a calculative power capable not only of arithmetic but also capable of, for instance, literary memory. I suddenly moved away. And indeed, did very well at school except for one subject, chemistry, which bored me. The master's way of teaching was the driest dust boys were ever subjected to. Moreover, he once, noticing me to be so inattentive, gave me six of the cane, which didn't improve matters at all. So I just let chemistry go, and so I didn't get the science prize. But I got every other prize that was to be got.' This breadth and enjoyment of intensive intellectual activity persisted beyond his school-days and into the present day. It is evident in his work as a creative mathematician, in his wide cultural interests, in his musical accomplishments, in his memory feats, in his sensitivity to nuances of conscious experience—as mentioned earlier, he is, calculative thinking aside, a rarely accomplished person.

It is noteworthy that this 'sparking off' experience was algebraic in nature and not merely arithmetical: he regards mental algebra as being on a much higher plane than mental arithmetic and incomparably more rewarding. But the effect of this experience was probably aided by two other circumstances. One was that, as a boy, he had no fear of arithmetic; for although it did not interest him then, he had often helped his father (an excellent arithmetician) with book-keeping after school. The other circumstance was the family tradition concerning his father's elder brother. This uncle 'was a farmer's son and it fell to him to work out volumes of timber and barrels, and so on. He had perhaps found in a book on mensuration Simpson's formula or variants of it and did them in his head. [Prof. Aitken's] father used to say that his uncle was the best arithmetician he had ever known by far and by far, and this legend or tradition was preserved in the family. [Prof. Aitken] supposed that was why when he became thirteen or fourteen he wanted to be something like his Uncle Tom' (Aitken, 1954, p. 307).

It would be wrong to regard this period of 'mental Yoga' as culminating merely in more rapid execution of elementary operations such as simple multiplication and division. This certainly happened: but repetitive drill was incidental to adventurous, persistent, and diversified intellectual exploration of both numerical relations and of his own calculative abilities; so there was also discovery of new and enlarging routines and of progressively higher-order relationships among numbers, and among classes of problems. A calculative routine, say for square-rooting, would be developed; it would be extended to further examples and, thereby, give rise to novel generalizations which, in their turn, would contribute new rules and procedures. New procedures and 'dodges' would then be combined together into yet more novel procedures of greater intricacy and compass for the solving of hitherto unmanageable problems or for the solving of old problems in more ingenious or more abbreviated ways.

A facet of this calculative exploration may be instanced. He developed left-to-right methods of multiplying, adding, and subtracting, as well as dividing. Numbers, if regarded as written sequences of digits, proceed from left to right. And for most calculative purposes, the important digits are those at the left-hand side. So it is 'natural' to treat each number by starting at its beginning. Furthermore, in mental calculation, there is less to remember by left-to-right methods than by the schoolroom methods of right-to-left (the greater memory work of the latter methods is typically managed by externalizing it into a written record). Usually, in left-to-right procedures, the outcome of each successive phase of calculation is progressively incorporated into a single result which approximates more and more closely to the final answer: there is no subsequent need to recall any early outcome once this incorporation has been made; and at a certain stage, the left-hand digits of the answer can be spoken before the entire calculation is complete. However, despite these advantages, left-to-right procedures may involve the thinker in complex co-ordinations. For example, in multiplying, the main product changes with each new partial product incorporated in it; so the attainment of the answer must necessarily copy behind the calculating, and the running memory span may have to be considerable. It is also necessary, as with all mental calculation, to remember the problem and to ensure the correctness of ongoing work because there is no written record of either the problem or the working.

In order to examine some features of left-to-right calculating, the writer asked some seventy students to work several multiplication problems by a method which, though not used by Professor Aitken, has clear affinities to some of his left-to-right plans. For example, the product of 123 and 456 would be attained as follows: $(100 \times 400) + (100 \times 50) + (100 \times 6) + (20 \times 400) + (20 \times 50) + (20 \times 6) + (3 \times 400) + (3 \times 50) + (3 \times 6)$. Each successive product is added to the cumulative sum of all earlier products, so that the answer proceeds from 40,000 to 45,000 to 45,600, and so on. The students found mental multiplication easier to do by this plan than by the mental use of the methods they habitually used when calculating with the aid of an ongoing written record. Yet even after an hour of self-paced practice, the students reported that difficulty or total failure could occur at any of many stages in the calculative sequence. They found difficulty in co-ordinating the subjectively distinct activities of: multiplying two numbers; adding two partial products; retrieving an already attained partial product; and recalling the stage now reached in the overall calculative plan. As practice continued, some students reported the development of occasional 'leaps', that is, an enlargement of the onleading correctly accomplished in a subjectively unitary 'step', e.g. proceeding from one partial product to the next without any awareness of the normally intermediate steps of specifying two numbers, multiplying them, and adding their product to the preceding partial product. However, even this economy raised difficulties for, lacking confidence in the correctness of the 'leap', the student might then undertake its verification and, thereby, disrupt the precarious continuity of the total thinking sequence. The experiences of these intelligent young adults point up something of the adventurousness, persistence, and cognitive capacity of the young Aitken in developing a rapidly effective usage of left-to-right methods.

Although Prof. Aitken cannot recall with certainty the details of his early calculative development, there was clearly that cumulative, hierarchically organized progression which so pervasively characterizes the acquisition of any comprehensive skill, e.g. learning to use a verbal language, to use telegraphic language, to typewrite, to play chess. 'In the process of constantly extending my ability, during the period when I was doing that, I had to think very much of what I was doing. I found that the gains were cumulative and, so to speak, stratified, in the sense that they formed a deposit sinking deeper and deeper into the subconscious and forming a kind of potential upon which, in certain states, I made drafts at astonishing speed.' With increasing speed, economy, and confidence, he learned how to proceed through relatively circumscribed transitions such as those from a particular number to its square, or from two particular numbers to their product (immediated recall?); likewise, how to proceed through less circumscribed transitions of a more general nature such as those from a problem to the most convenient method of solving it (intuitive judgement?). The gross developmental outcome of all this intellectual effort and synthesis was to carry the young Aitken outwith the normal range of mental calculators. Figuratively speaking, he built up an extensive organized library of calculative resources around which he could find his way with ingeniously methodical rapidity: he could, so to speak, skim round the appropriate sequence of files pulling out apposite calculative instructions of varying levels of generality while, at the same time, implementing these instructions as they come to hand and so progressively specifying the co-ordinates of what is required by way of a confidently correct answer—all this before most people have taken in what the problem is.

After the age of about thirty-five, practice in mental calculating was much less intensive than it had been. The result was that the skill deteriorated. Failure to maintain his skill at concert pitch was due, in part, to increasing demands made on his time by other matters and, in part, to lessened interest in mental calculation as such. 'When I came from New Zealand [in 1923] and first used an arithmometer, even of the antiquated types then available, I saw at once how useless it was, how gratuitously useless, to carry out for myself any mental multiplication of large numbers. Almost automatically I cut down my faculty in that direction, though I still kept up squaring and reciprocating and square-rooting, which have a more algebraic basis and a statistical use. But I am convinced that my ability deteriorated after that first encounter' (Aitken, 1954, p. 303). In addition to lack of intensive and regular practice, he also feels that he is 'no longer in prime condition with regard to the sheer physical stamina' necessary for those elaborate calculative feats (such as squaring a ten-digit number) which gave him pleasure in his late teens and early twenties. 'You really have to be in something like perfect athletic form' to do this well. So his present-day skill derives from the more highly developed skill of early adulthood as maintained by relatively infrequent practice. And present-day practice is, in contrast to the 'mental Yoga' of earlier years, not engaged in primarily for its own sake. It is incidental to his mathematical teaching and research, and especially to the pursuit of those parts of higher algebra where conjectured theorems may be tested by actual numerical examples. 'There, if several examples prove to give the result which one expects from the theorem, there is the incentive to go ahead with the purely theoretical proof; and the quickest way of disproving a conjectured theorem is to try it out numerically and to find that it is incorrect in some trial examples.' So he is not, as he formerly was, interested in mental calculating for the sake of its challenging length or complexity but more for its utility in the subtler tasks of solving theoretical and practical problems involving number.

IV. Experiential characteristics of mental calculating

Prof. Aitken is not able to give such an account of his experiencing during calculation as would enable the present writer to empathize entirely with him: this writer does not fully know what it would feel like to be Prof. Aitken doing a calculation. Reports on the experiential aspects of calculating can be presented under three headings.

(i) *Non-sensory nature of 'numbers'*. He does not calculate by manipulating experienced representations of numbers which have any distinct degree of sensory realism. There is certainly no visual imaging, nor does there seem to be any auditory or kinaesthetic imaging (all three modes of imaging are familiar to him in other, non-calculative contexts). He reports that he could calculate in visual or auditory terms but that this would greatly slow him down. His complexly constrained, intricately timed onleading is, then, unencumbered by vivid imaging and does not, to any reportable degree, involve sensory-type representations of numbers. His onleading is more economical than this; it is further removed from the cumbersome, developmentally primitive level of sensory-motor activity. His calculative coordinations are more akin to those involved in such activities as sprinting, boxing, typewriting, writing, and playing fast ball games. Champion skiers, swimmers and dancers do not perform by consciously representing either limb movements or the many exigencies which influence their activity from moment to moment. Master instrumentalists, singers, and composers do not execute their skills by manipulating sensory-type representations of musical notes. Fluent conversationalists do not emit their sequences of talk by juggling with experienced representations of words. Neither does Prof. Aitken calculate by manipulating 'sensory numbers': he proceeds with a paucity of sensory awareness and his proceeding issues, at the appropriate time, into vocalizing. From moment to moment, he intends in schematic fashion to *do* this or that. He can describe what he intends to do but, like the fluent talker or typist, he cannot give what an observer would regard as an adequate description of the intending as such.

(ii) *The main calculative stream.* During calculation, he is not an automaton lacking self-awareness. 'I must be relaxed, yet possessed, in order to calculate well. I believe that conscious and subconscious activities are conspiring or in rapid alternation. I seem to move on several different levels. And last of all, when the result is complete, I return to the normal level of ordinary social contact.' In this state of relaxed absorption, he is acutely aware of what might be called the main stream of calculative activity and also of the often reported side-flashes. However, he cannot describe the nature of this awareness, even though he can describe his doings by translating them into a socially shared language which has words and symbols for numbers, for methods, and for operations on numbers. This socially shared language is, of course, not used in actual calculation; if it were, the whole activity could not proceed with its characteristic rapidity. Social language is only used afterwards as a means of describing to other people what was done. Like all descriptive languages, it is selective and, in some measure, misleading: 'description to others always modifies somewhat the actual sequence of events'.

(3) *Onleading by leaps.* Much of his thinking involves leaping, in the sense of onleading which is accomplished without any involvement of conscious mediation or conscious lapse of time but yet onleading which, if made by less expert thinkers, would involve elaborate and time-consuming successions of try-and-check activities. For example, with this thinker, as with many other people, 12 is the immediated product of 3 and 4: but unlike most people, the transition from '9 times 12,345' to '111,105' is also immediate for this thinker. Consider also his 'simply seeing in one go' the number 1961 as 37 times 53, and 44 squared plus 5 squared, and 40 squared plus 19 squared. Other leaps concern procedural judgements, that is, diagnosing what method is best to use in calculation. These high-level procedural diagnoses derive from a breadth of past experience which is fully comparable to (and possibly in excess of) that which lies behind the so-called position sense of the chess-master, or the swiftly impressionistic diagnoses made by some experienced physicians, or the intuitive snap judgements made by experts in many fields of science, art, and commerce. Yet other leaps occur between attaining an answer and recognizing either that it is correct or that something is wrong. 'I find that if I do not doubt the result of a calculation, it is usually always correct; but if I have any residual doubt, then some correction is usually required.'

One last introspective report is worth mentioning. When he attains the answer to a problem with rapidity, good timing and a feeling of 'all correct', then he cannot easily say whether he calculated this answer or recalled it—especially if he definitely knows that he has done this particular calculation before. In other words, the activity may lack experiential characteristics enabling him to apply to it one or other of two mutually exclusive labels. This report is a reminder of a familiar fact: people classify one onleading sequence of activity as 'thinking' and another as 'recalling' and do so by criteria which are not entirely hard-and-fast: these two broad categories, useful though they are, shade into each other. This being so, it is not surprising that Prof. Aitken should sometimes find it impossible to distinguish between recalling and thinking. What is striking is that, in his case, the vague borderline between recalling and calculating should straddle a complexity of achievement which, for most people, so evidently implies hard thinking.

V. An example of numerical recalling

Prof. Aitken is capable of many remarkable feats of long-term memory for personal experiences, music, and verbal material in English and Latin (e.g. the writer has been able to check the complete accuracy of a list of twenty-five unrelated words which Prof. Aitken recalled from a brief memorizing session in the Edinburgh Psychology

Department of more than a quarter of a century before). With numerical material, one of his memory feats is the rapid recitation of the value of Pi to a thousand places or more. The characteristics of this recitation are noteworthy because of the intensive and rhythmic co-ordination involved and also the changes produced in this co-ordination when recitation proceeds backwards.

Pi is the ratio of the circumference of a circle to its diameter; it is a non-recurring decimal of great mathematical importance. Some years ago Prof. Aitken memorized this decimal to a thousand places. He did it as a kind of intellectual *tour de force* and acknowledges that it 'would have been a reprehensibly useless feat had it not been so easy'. He ranged the digits in rows of fifty, each fifty being divided into ten groups of five. He then found that if he read over the digits in a particular rhythm and tempo, he could fairly easily memorize a group of fifty digits. 'The learning was rather like learning a Bach fugue.' Apart from this rhythm and tempo, there was no 'interpretation' of the digits, no devising of mnemonic relations. 'Mnemonics I have never used and deeply distrust. They merely perturb with alien and irrelevant association a faculty that should be pure and limpid. Our present civilization, not only urban but rural, full of noise and interruption as it is, offers every hindrance to that relaxed meditation upon which the strength of memory thrives best' (Aitken, 1954, p. 301).

As a result of this learning, and periodic 'brushing up', he is able to recite the first thousand digits of Pi and also able to pick up the digits at any given point within the sequence without having to start at the beginning. The writer has tape-recorded this forward recitation, checking it against a typed copy of the digits, and the following report is based on this recording.

Sitting relaxed and still, he speaks the first 500 digits without error or hesitation. He then pauses, almost literally for breath. The total time taken is 150 sec. The rhythm and tempo of speech is obvious; about five digits per second separated by a pause of about $\frac{1}{2}$ sec. The temporal regularity is almost mechanical; to illustrate, each successive block of fifty digits is spoken in exactly 15 sec.

The second 500 digits are also errorless but there are hesitations and even a few self-corrections. The basic tempo and rhythm is evident but occasional haltings and fumblings give rise to temporal irregularity. Thus, the time for each successive block of fifty digits is 15, 21, 29, 25, 35, 15, 18, 30, 21, and 18 sec., respectively. The total time is 227 sec. This less secure recall is partly the result of fatigue, for he admits that recalling is a strain. Commenting on this recall, Prof. Aitken wrote as follows. 'The first 550 digits were no trouble. Then I think I ran into some trouble for the following reason. Before the days of computing machines there was a kind of competition between calculators (human, I mean) in seeing how far they could calculate Pi. In 1873, Shanks carried this to 707 decimals; but it was not till 1948 that it was discovered that the last 180 of these were wrong. Now, in 1927 I had memorized those 707 digits for an informal demonstration to a students' society, and naturally I was rather chagrined, in 1948, to find that I had memorized something erroneous. When Pi was re-calculated to 1000 and indeed more decimals, I re-memorized it. But I had to suppress my earlier memory of those erroneous digits, 180 of them; and this, I think, sufficiently accounts for a certain tentativeness that seizes me in that region from about the 551st to the 751st digit. But fatigue had doubtless something to do with it also; and I am inhibited, too, by feeling how useless the feat is—to have written twelve bars of good music would be far more rewarding.'

He is then asked to recite the last fifty digits backwards and does so correctly in a total time of 34 sec. This recitation is still distinctly grouped into blocks of five digits but the whole activity is slower than before and there are some hesitations. In subsequent discussion, the procedural difference between forward and backward recall emerges. In forward recall, no visual imaging occurs. 'Seeing would put me off. If I were forced to visualize [either here or in calculation], I would be much slower. The strong auditory-rhythmic pattern carries almost like bars of music. It's this funny faculty of neither seeing or hearing.' In backward recall, visualizing is employed. 'I bring the numbers along in blocks of five, and manage to see them as a whole, and read them backwards. But that is very much contrary to my usual practice and it goes against the grain.' Visualizing is also used when he is given a string of digits to memorize and is recalling them either backwards or in some order other than the straight forward order (e.g. in the number-square task where a succession of digits are entered horizontally into the cells of an imaginary square and then recalled vertically or diagonally). In the present study, these are the only performances which he finds it necessary to organize in a way which involves distinct visual imaging.

VI. Postscript

This study has attempted a naturalistic exploration of one class of activities in one exceptional man, and has succeeded in describing some fragmentary aspects of those activities. It now seems fitting to conclude with a comment from the man whose thinking was studied—a comment which pinpoints the main shortcoming of the whole study. 'However much analysis may seem to lay bare the concurrent activities, they are not explained until they are synthesized again and integrated, and then indeed the analysis is seen to be deceptive and defective to that extent. This, of course, is true of all analysis, psychological or physical. "We murder to dissect", said Wordsworth; or phrased otherwise, the whole is ever so much more than the sum of its parts.'

The writer cannot fully express the extent of his gratitude to Prof. Aitken. He acted as subject and answered a host of questions. He commented on successive drafts of the present paper, and permitted its publication despite the personal nature of its contents. His friendly and patient collaboration provided the writer with uniquely first-hand insights into the intricate texture of highly proficient thinking.

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